Profesor:
Jonathan Cumpa Velásquez

TRIGONOMETRÍA

GRUPO PITÁGORAS

CIRCUNFERENCIA TRIGONOMÉTRICA

DEFINICIÓN Y REPRESENTACIÓN DEL SENO, COSENO Y TANGENTE

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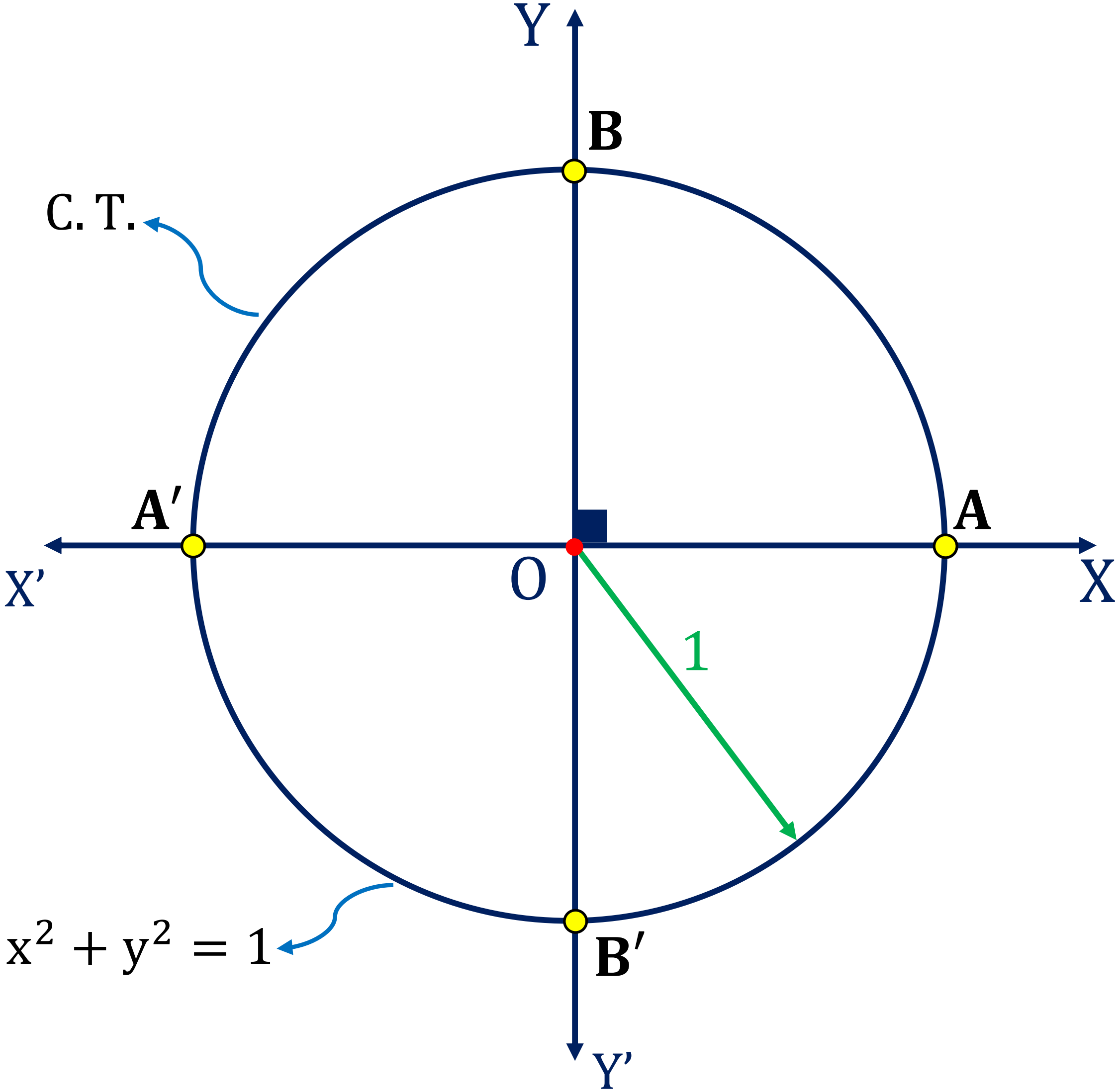
1)Definición:



- ✓ $A(1;0)$: Origen de arcos
- ✓ $B(0;1)$: Origen de complementos
- ✓ $A'(-1;0)$: Origen de suplementos
- ✓ $B'(0;-1)$: No tiene nombre específico

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Ejemplo:

Si el punto $\left(\frac{x}{10};\frac{x+2}{10}\right)$ pertenece a la C. T., calcular x, además se sabe que "x" es negativo.

Resolución:

Si un punto $P(a;b) \in C.T \rightarrow$ satisface la ecuación: $x^2+y^2=1$

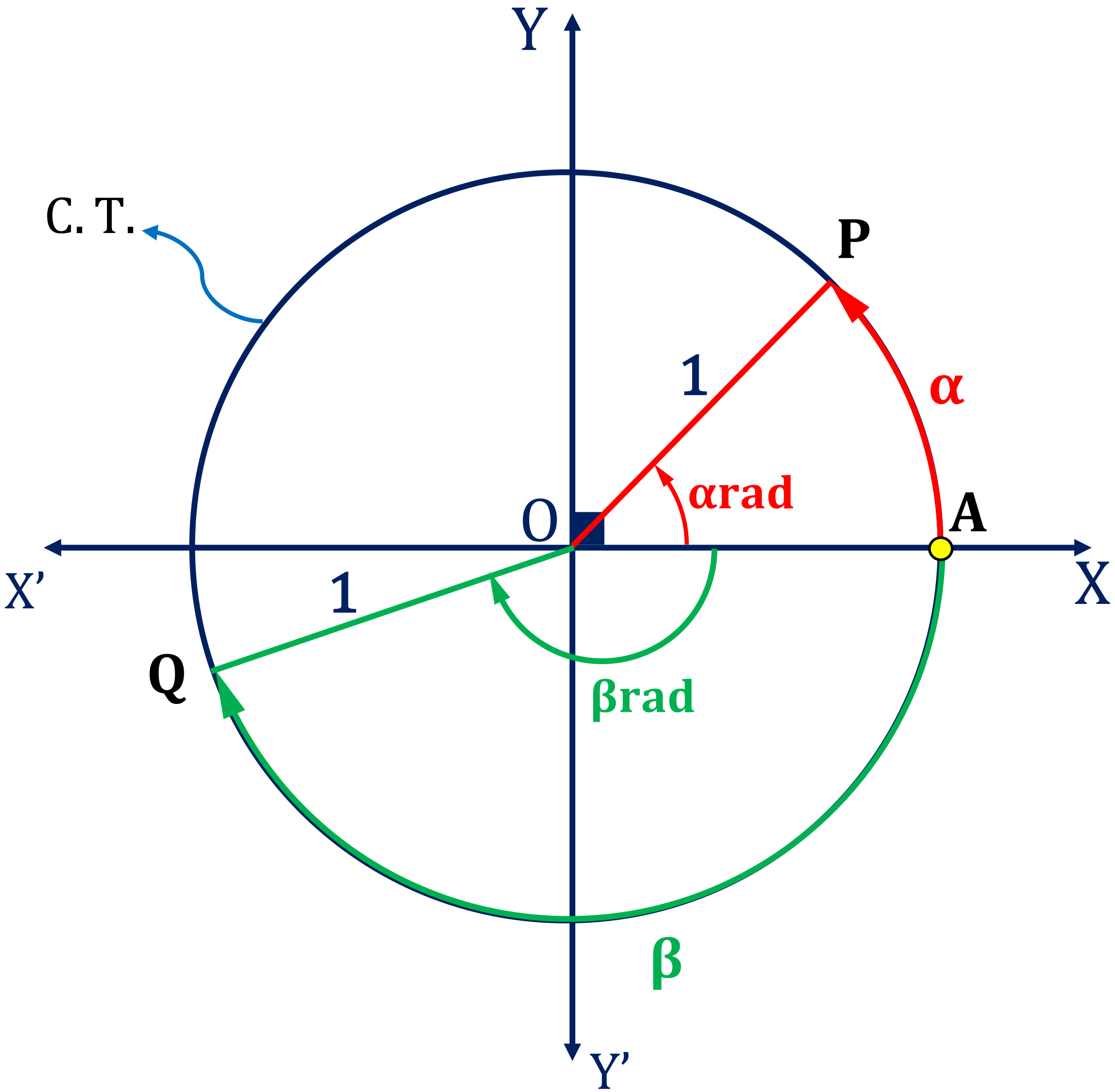
$$\left(\frac{x}{10}\right)^2+\left(\frac{x+2}{10}\right)^2=1$$

$$(x)^2+(x+2)^2=100$$

$$x^2+2x-48=0$$

$$\boxed{x=-8} \lor x=6$$

2)Arco en posición estándar:



- En el sector circular POA

$L = \theta . R$

$\hookrightarrow L_{\widehat{AP}} = \alpha . 1$

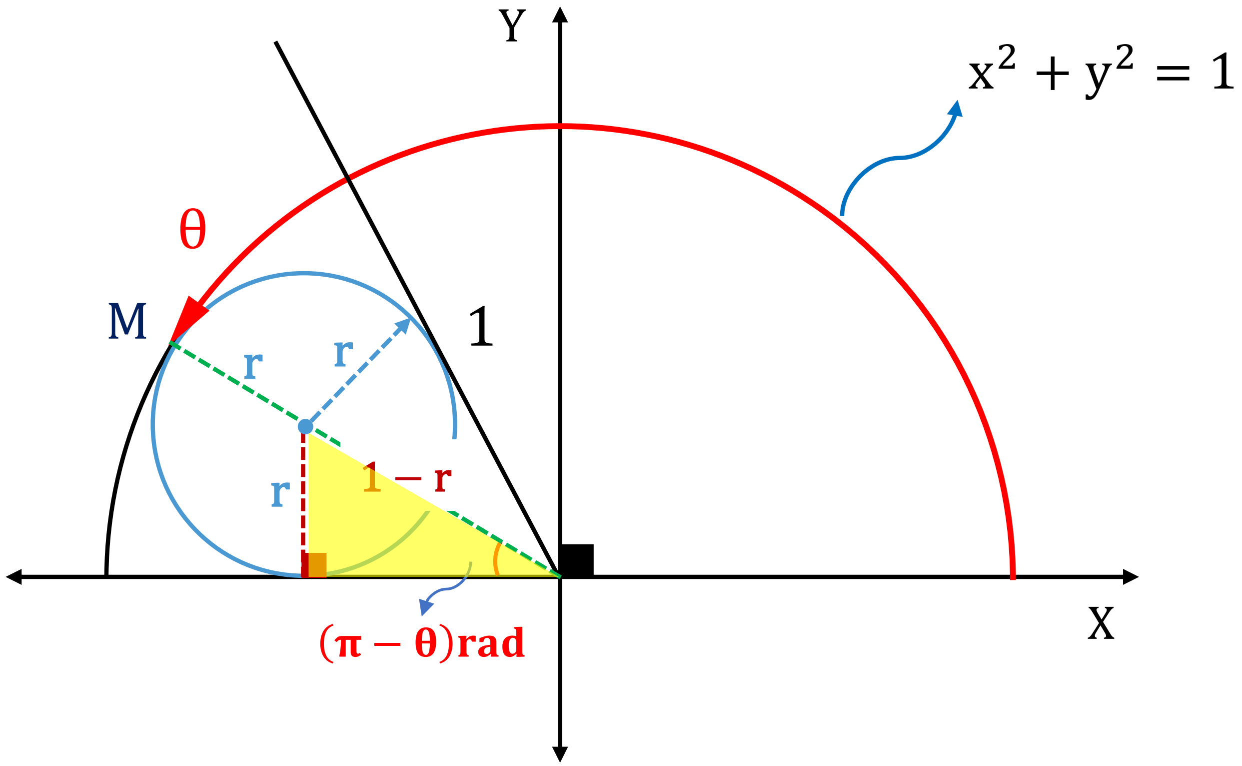
$L_{\widehat{AP}} = \alpha$

- $m\sphericalangle\alpha rad > 0 \;\rightarrow\; \alpha > 0$
- $m\sphericalangle\beta rad < 0 \;\rightarrow\; \beta < 0$

Ejemplo:

En la figura, calcular “r” en función de "θ“ (M: Punto de tangencia)

Resolución:



- De la figura:

$$Csc(\pi - \theta) = \frac{1 - r}{r}$$

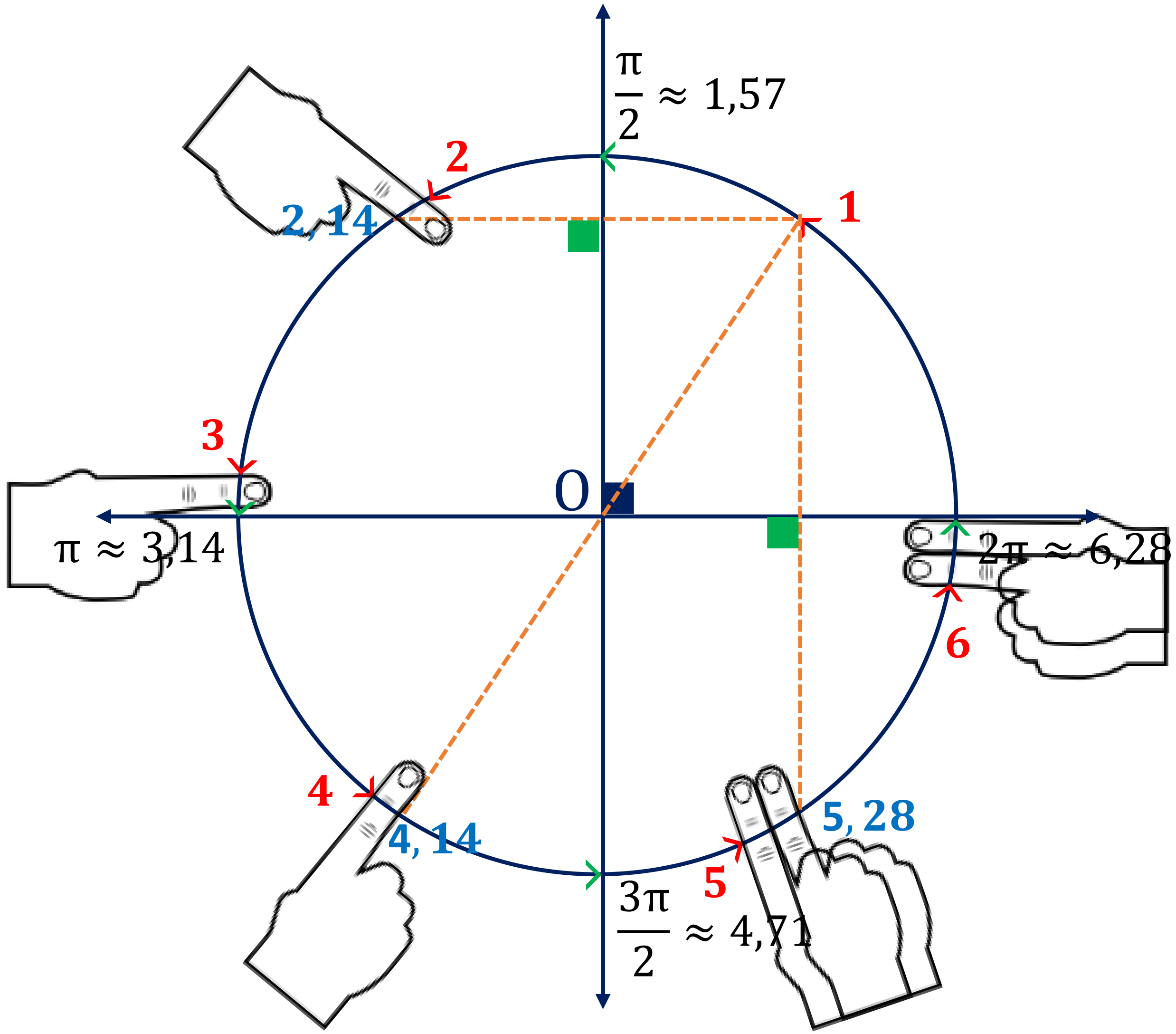
$$Csc\theta = \frac{1 - r}{r}$$

$$Csc\theta = \frac{1}{r} - 1$$

$$Csc\theta + 1 = \frac{1}{r}$$

$$\therefore \mathbf{r} = \frac{\mathbf{1}}{\mathbf{Csc\theta + 1}}$$

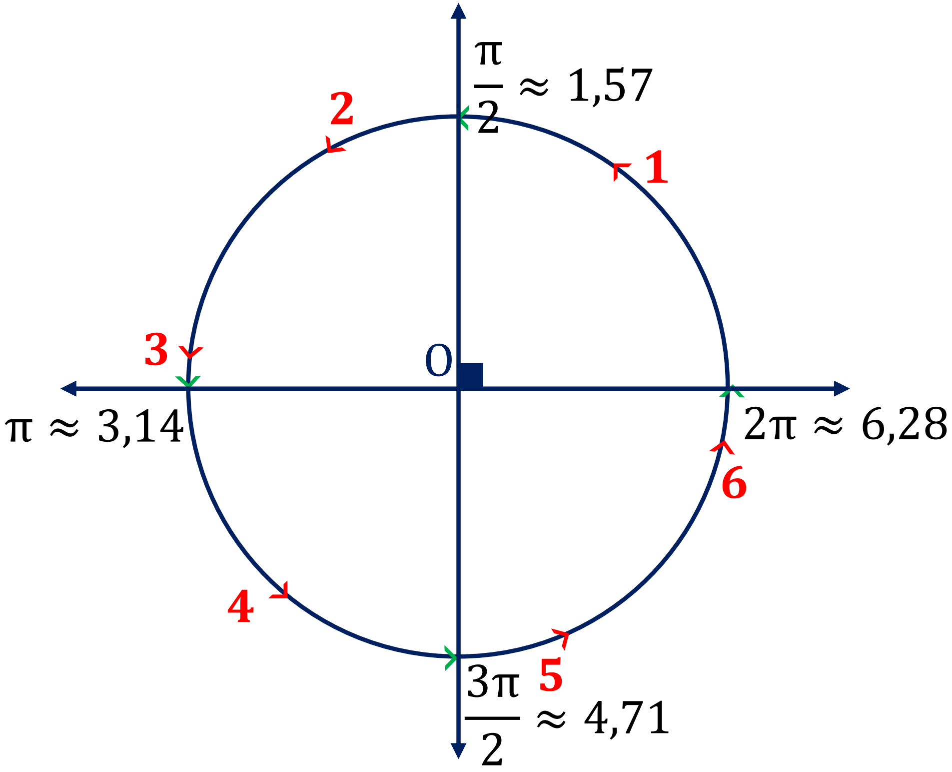
2.1) Extremos numéricos enteros de un arco:



Ejemplo:

Hallar el signo de las expresiones: Sen2; Cos3; Tan5, Cot4, Sec1, Csc6

Resolución:



Sen2: $(+)$

Cos3: $(-)$

Tan5: $(-)$

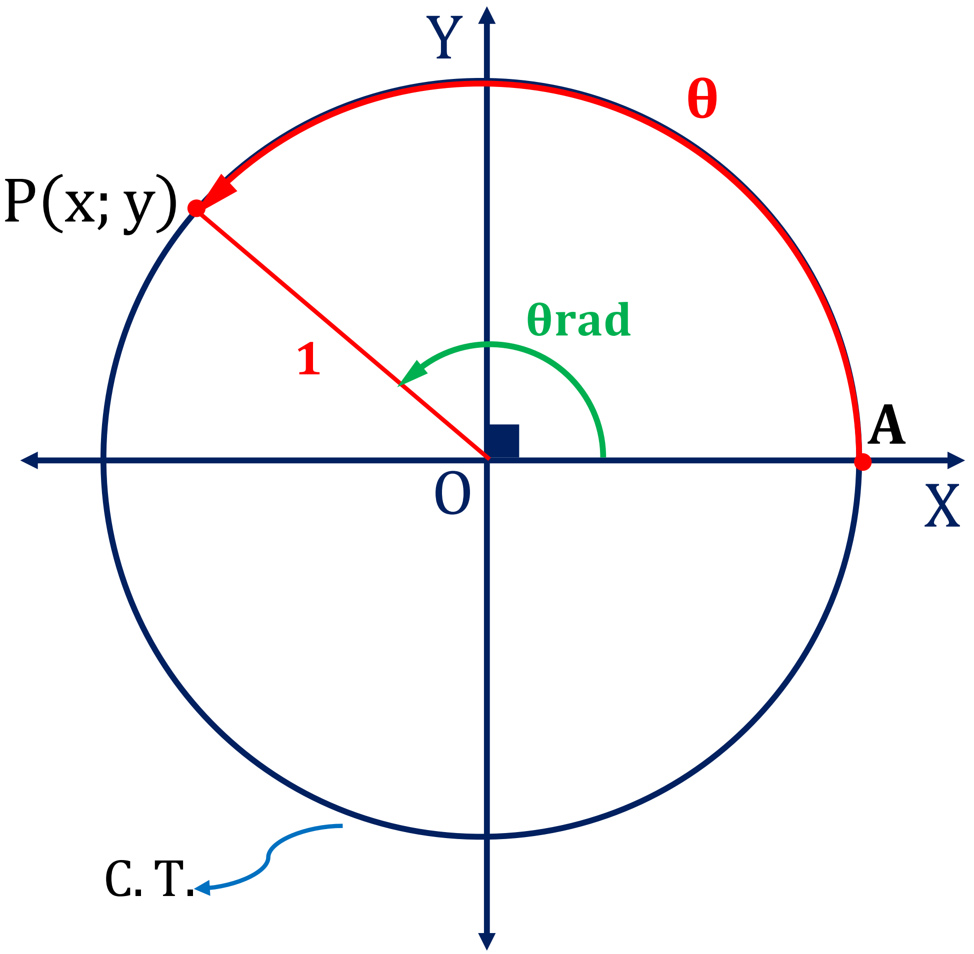
Cot4: $(+)$

Sec1: $(+)$

Csc6: $(-)$

3)Razones trigonométricas de un arco en posición estándar:

$$\mathbf{RT(arco) = RT(\sphericalangle Central)}$$



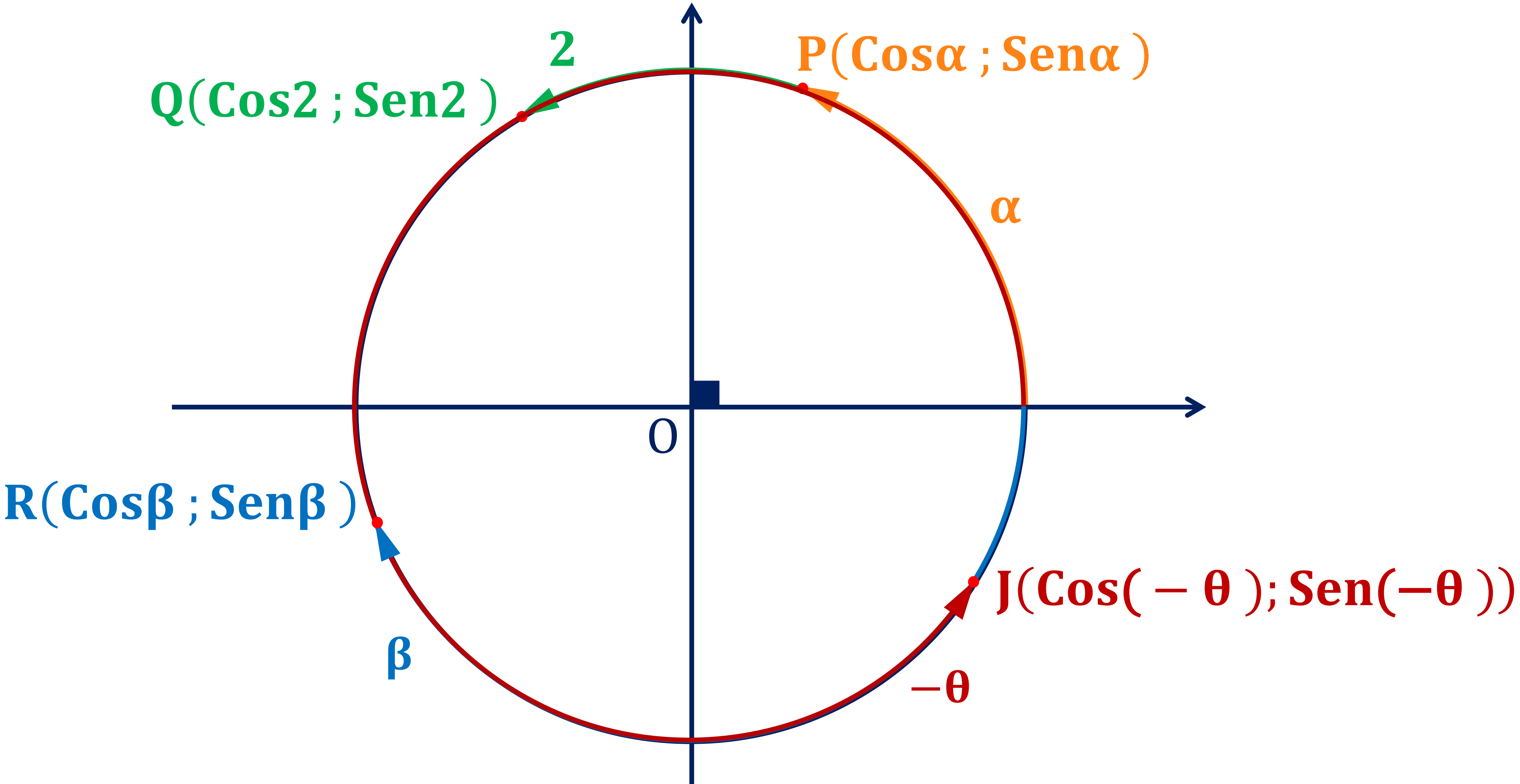
$$\mathrm{Sen}(\theta) = \mathrm{Sen}(\theta\mathrm{rad}) = \frac{y}{r} = \frac{y}{1} = y \longrightarrow \mathrm{Sen}\theta = y$$

$$\mathrm{Cos}(\theta) = \mathrm{Cos}(\theta\mathrm{rad}) = \frac{x}{r} = \frac{x}{1} = x \longrightarrow \mathrm{Cos}\theta = x$$

$$\mathrm{Tan}(\theta) = \mathrm{Tan}(\theta\mathrm{rad}) = \frac{y}{x} = \frac{\mathrm{Sen}\theta}{\mathrm{Cos}\theta} \longrightarrow \mathrm{Tan}\theta = \frac{\mathrm{Sen}\theta}{\mathrm{Cos}\theta}$$

3.1)Coordenadas del extremo de un arco:

$$\mathbf{P(x;y) = P(Cos(arco);Sen(arco))}$$

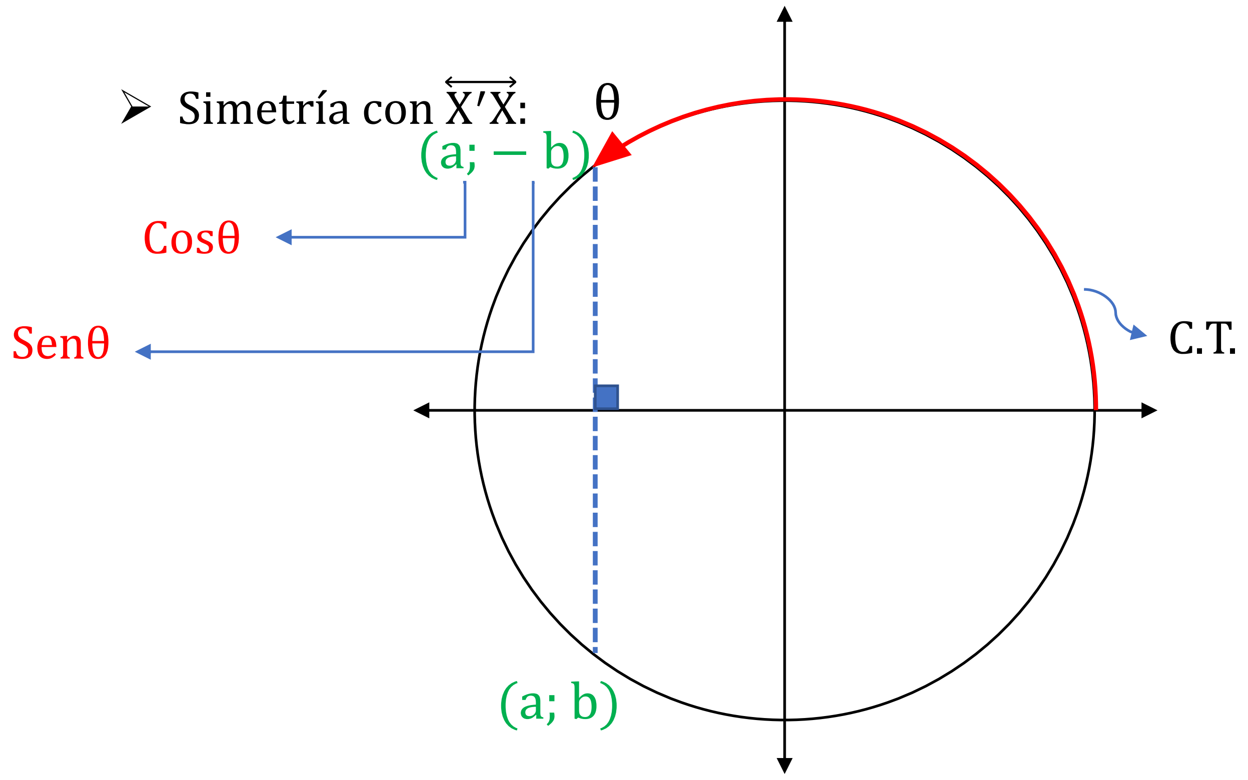


Ejemplo:

Del gráfico mostrado, la expresión :

$$a\mathrm{Tan}\theta + b\mathrm{Csc}\theta + 1$$

equivale a :



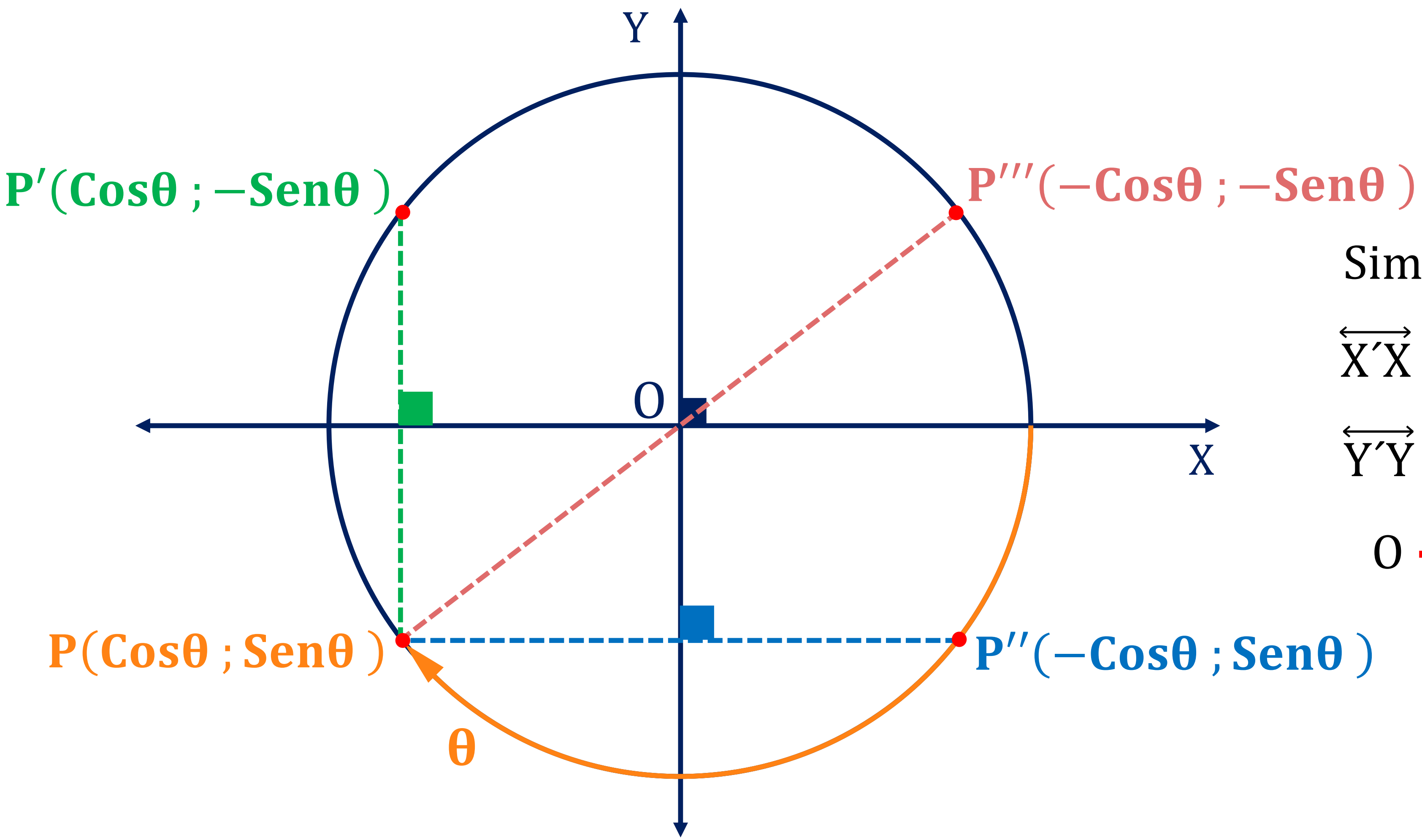
Resolución:

$$A = a\mathrm{Tan}\theta + b\mathrm{Csc}\theta + 1$$

$$A = \cancel{\mathrm{Cos}\theta} \times \frac{\mathrm{Sen}\theta}{\cancel{\mathrm{Cos}\theta}} + \cancel{(-\mathrm{Sen}\theta)} \times \frac{1}{\cancel{\mathrm{Sen}\theta}} + 1$$

$$\mathbf{A = Sen\theta}$$

3.2)Coordenadas simétricas:



Simetría con:

$\overleftrightarrow{X'X}$ ⟶ "y" cambia de signo

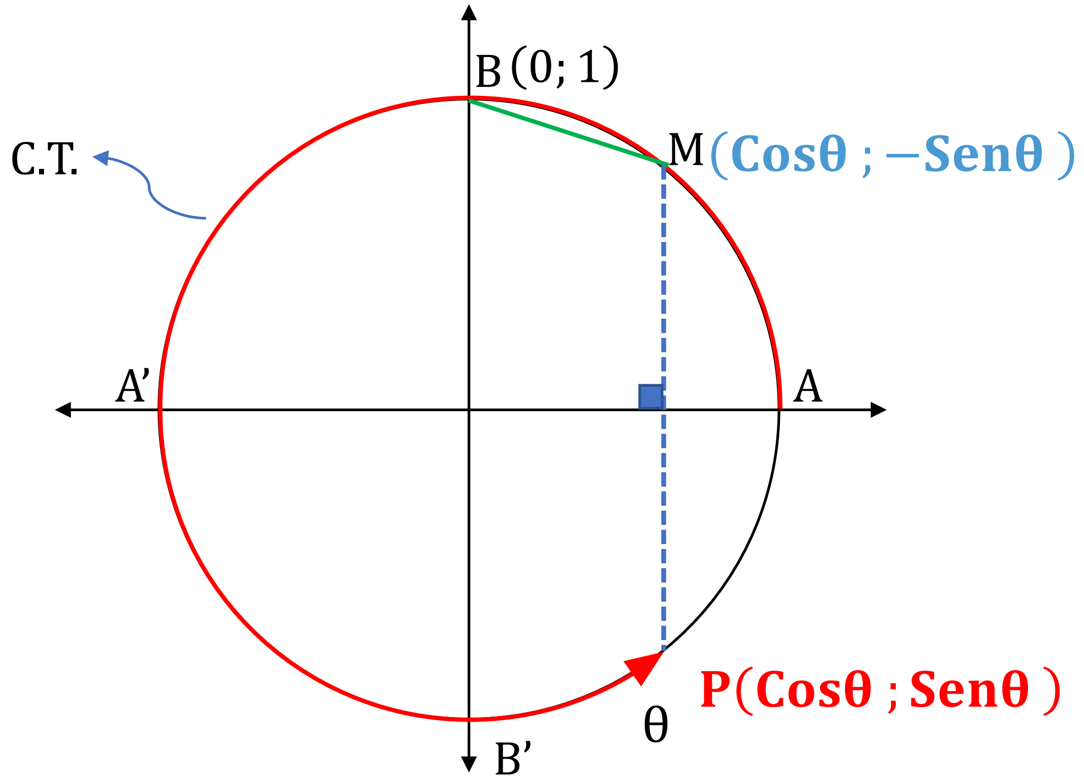
$\overleftrightarrow{Y'Y}$ ⟶ "x" cambia de signo

O ⟶ "x e y" cambian de signo

Ejemplo:

De la C.T. determinar el valor del segmento BM en términos de θ.

Resolución:



$$BM = \sqrt{(0 - \text{Cos}\theta)^2 + (1 - (-\text{Sen}\theta))^2}$$

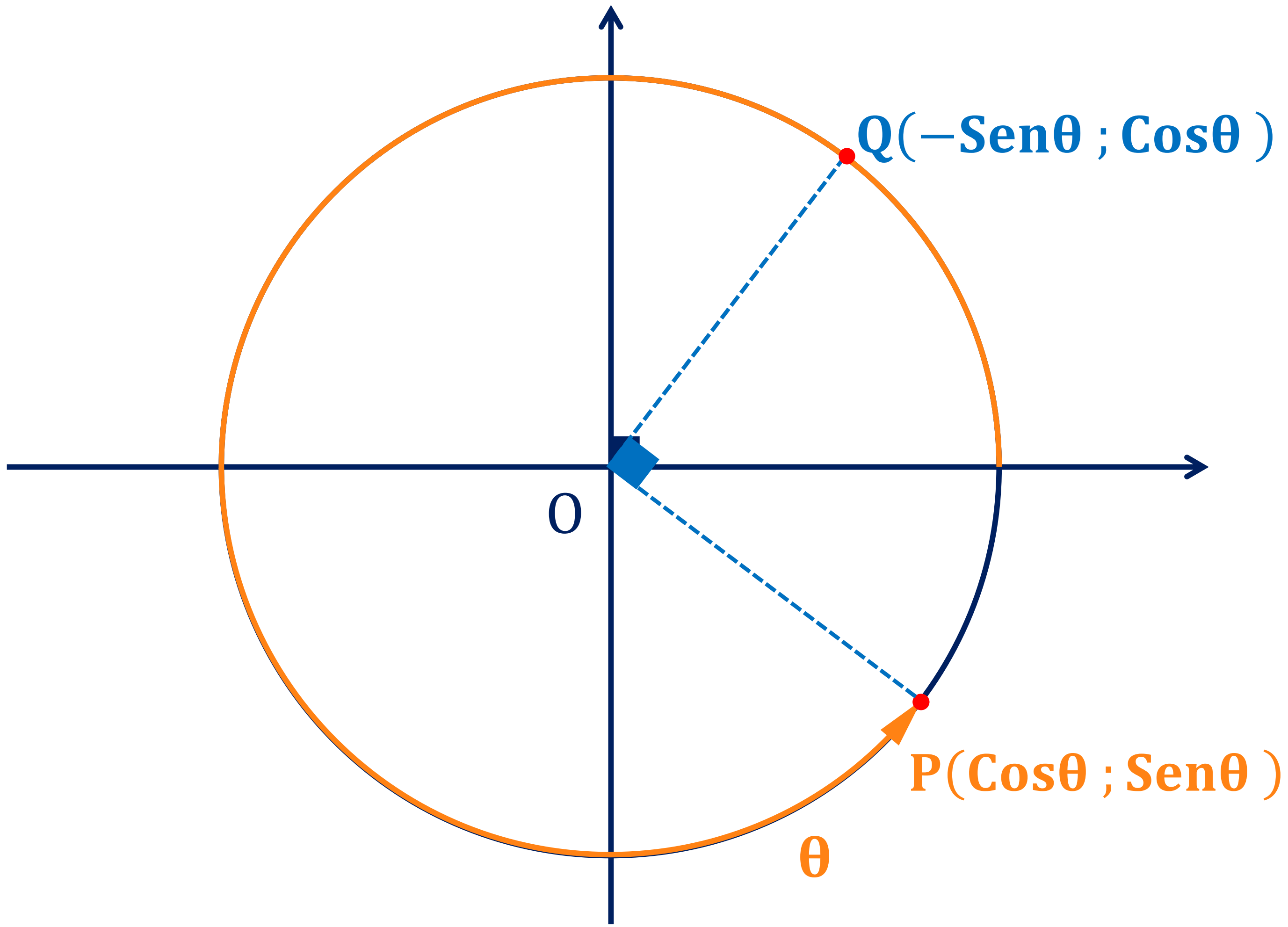
$$BM = \sqrt{\text{Cos}^2\theta + (1 + \text{Sen}\theta)^2}$$

$$BM = \sqrt{\text{Cos}^2\theta + 1 + 2\text{Sen}\theta + \text{Sen}\theta^2}$$

$$BM = \sqrt{2 + 2\text{Sen}\theta}$$

$$\therefore \overline{BM} = \sqrt{2(1 + \text{Sen}\theta)}$$

3.3)Coordenadas ortogonales:



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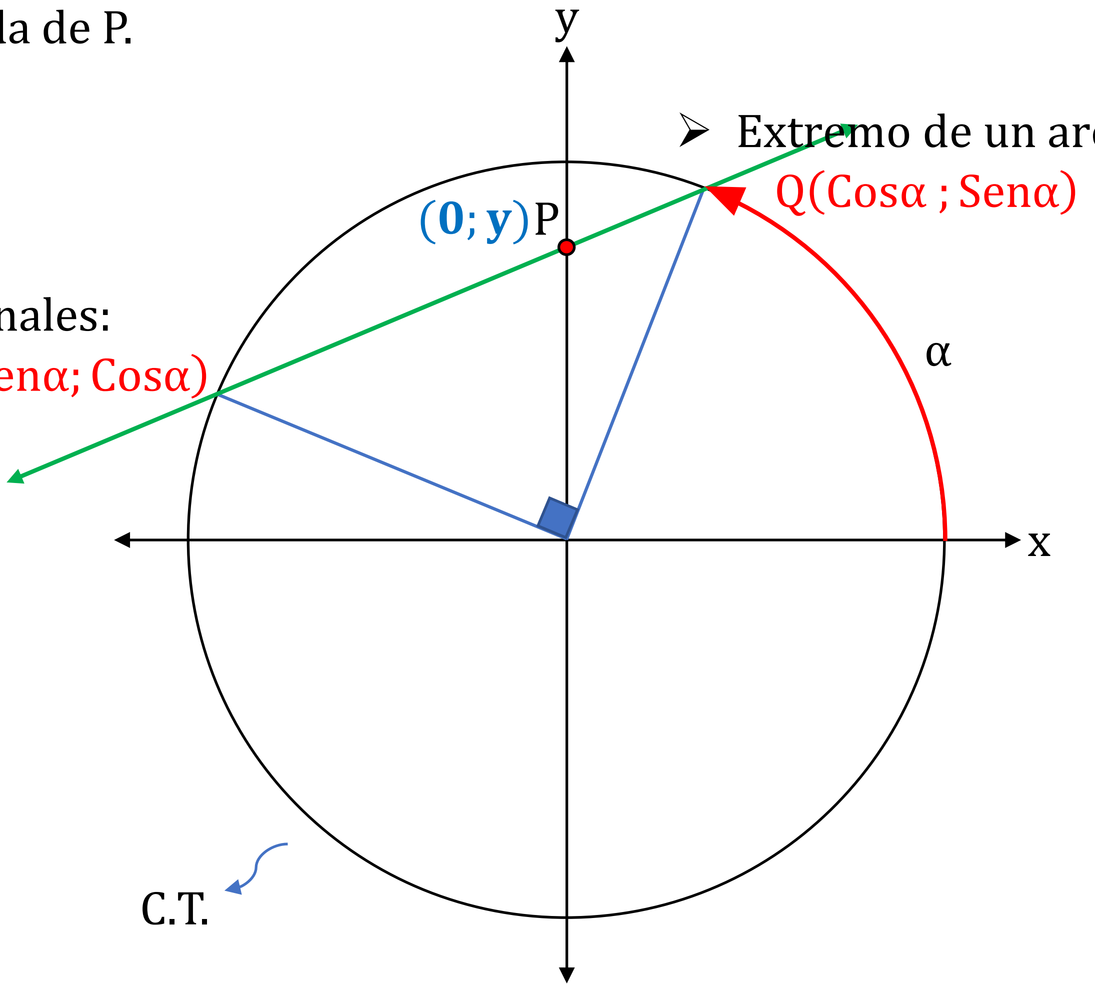
Ejemplo:

Determinar la ordenada de P.

- Coordenadas Ortogonales:
- Extremo de un arco:



Resolución:

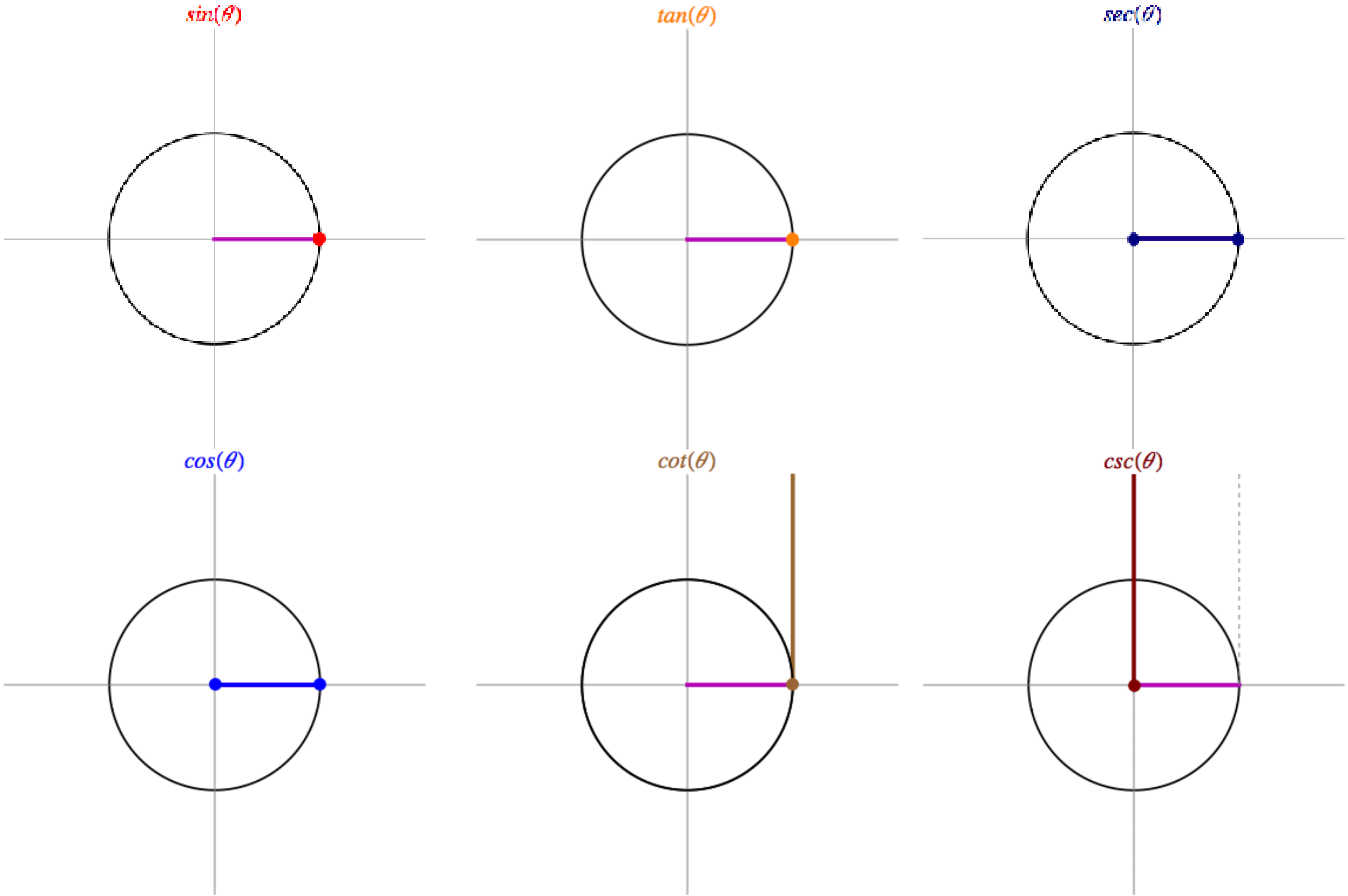
- $m_{Q\,y\,R} = m_{P\,y\,R}$

$$\frac{\text{Sen}\alpha - \text{Cos}\alpha}{\text{Cos}\alpha - (-\text{Sen}\alpha)} = \frac{y - \text{Cos}\alpha}{0 - (-\text{Sen}\alpha)}$$

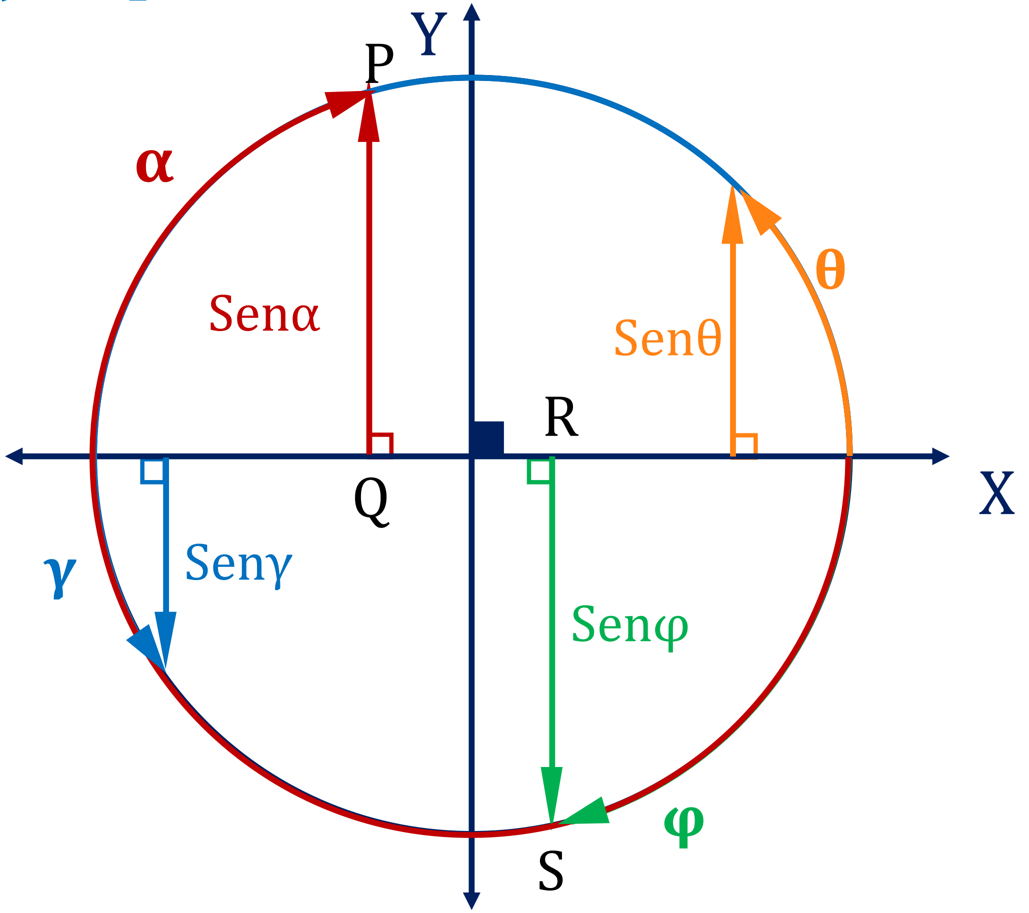
$$\frac{\text{Sen}\alpha^2 - \text{Sen}\alpha\text{Cos}\alpha}{\text{Cos}\alpha + \text{Sen}\alpha} + \text{Cos}\alpha = y$$

$$\therefore \mathbf{y} = \frac{\mathbf{1}}{\mathbf{Sen\alpha + Cos\alpha}}$$

4)Líneas trigonométricas:



4.1) Representación de la línea Seno:



i. Valor $\begin{cases} (+): \text{Sen}\theta \\ (-): \text{Sen}\gamma \\ 0 \;: \text{Sen}\pi \end{cases}$

ii. Longitud: $(+)$, $|\text{Seno}|$

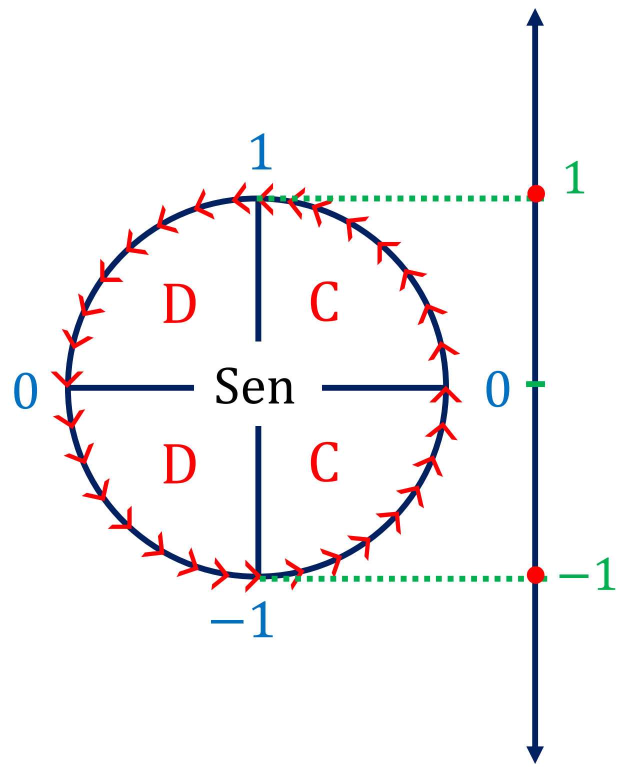
$$\overline{PQ} = |\text{Sen}\alpha| = \text{Sen}\alpha$$

$$\overline{RS} = |\text{Sen}\varphi| = -\text{Sen}\varphi$$

✓ $\alpha < \varphi < \theta < \gamma$

✓ $\text{Sen}\varphi < \text{Sen}\gamma < \text{Sen}\theta < \text{Sen}\alpha$

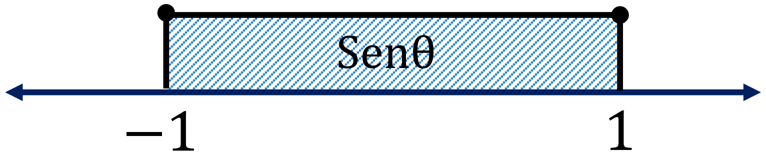
4.1.1) Variación analítica :



i. Extensión:

$\forall\ \theta \in \mathbb{R}$

$-1 \leq \text{Sen}\theta \leq 1$



- Si: $\theta \in \text{IC}$

  $\mathbf{0 < Sen\theta < 1}$

- Si: $\theta \in \text{IIC}$

  $\mathbf{0 < Sen\theta < 1}$

- Si: $\theta \in \text{IIIC}$

  $\mathbf{-1 < Sen\theta < 0}$

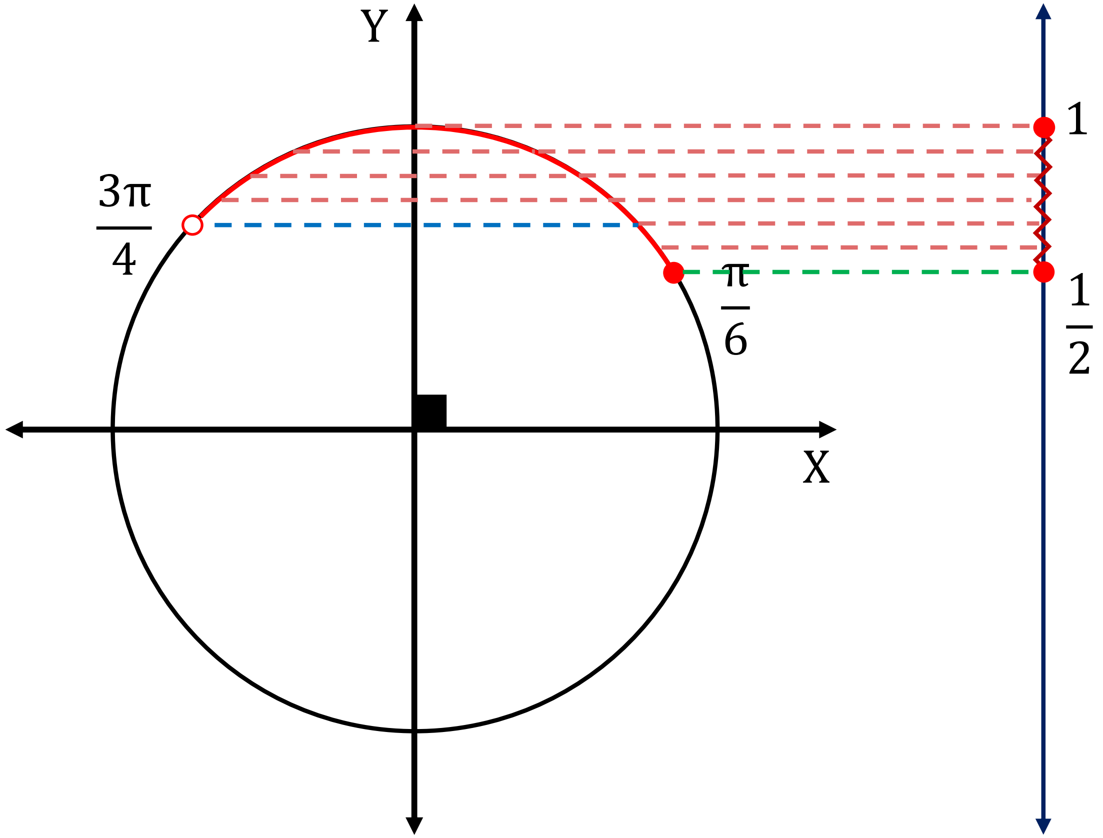
- Si: $\theta \in \text{IVC}$

  $\mathbf{-1 < Sen\theta < 0}$

Ejemplo:

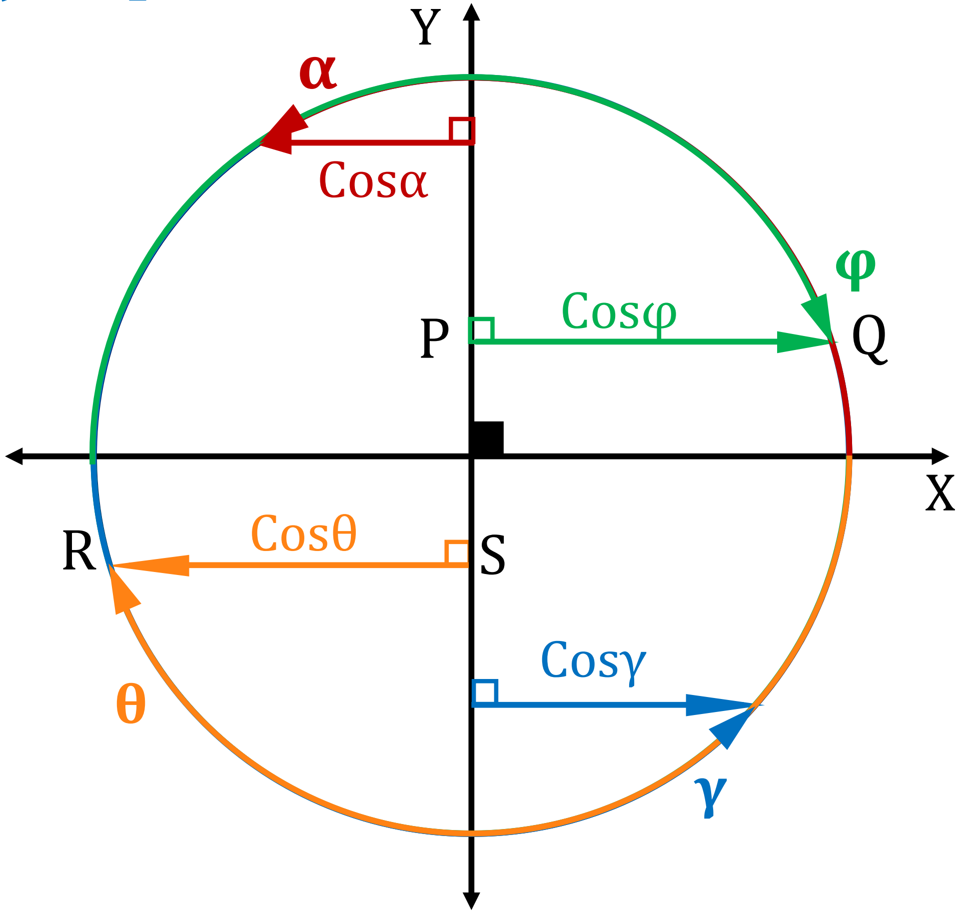
Determinar la extensión del Senθ si $\theta \in \left[\frac{\pi}{6};\frac{3\pi}{4}\right[$

Resolución:



$\therefore$ **Senθ** $\in \left[\frac{1}{2};1\right]$

4.2) Representación de la línea Coseno:



i. Valor $\begin{cases} (+): \cos\gamma \\ (-): \cos\alpha \\ 0 \ : \cos\frac{\pi}{2} \end{cases}$

ii. Longitud: $(+)$, $|\text{Coseno}|$

$$\overline{PQ} = |\cos\varphi| = \cos\varphi$$

$$\overline{RS} = |\cos\theta| = -\cos\theta$$

✓ $\varphi < \theta < \alpha < \gamma$

✓ $\cos\theta < \cos\alpha < \cos\gamma < \cos\varphi$

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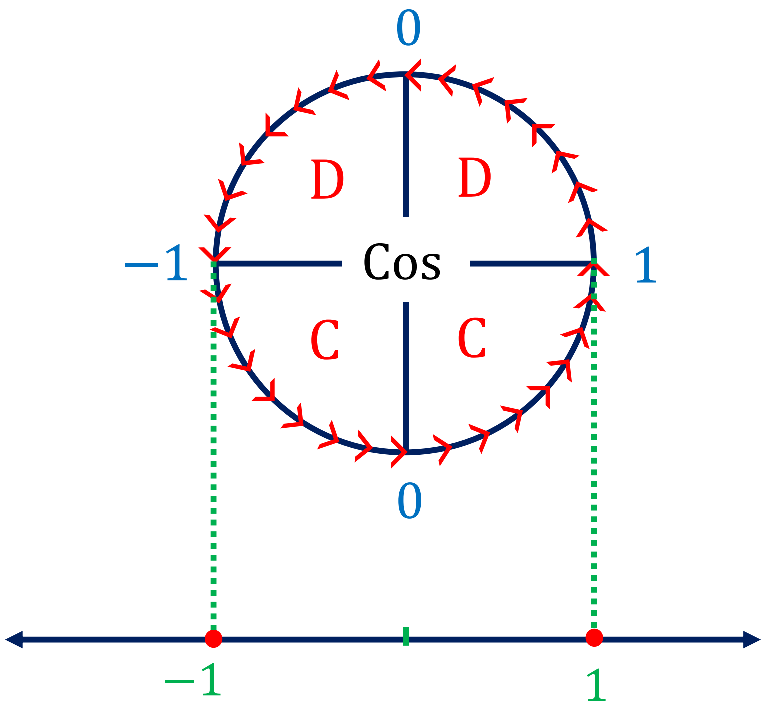
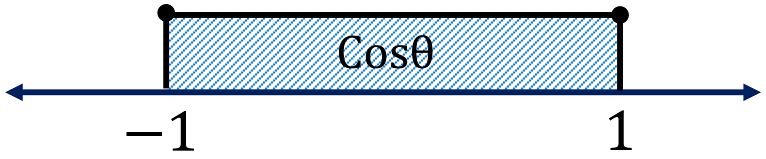
4.2.1)Variación analítica :



i. Extensión:

$\forall\ \theta \in \mathbb{R}$

$-1 \leq \mathrm{Cos}\theta \leq 1$



- Si: $\theta \in \mathrm{IC}$

  $\mathbf{0 < Cos\theta < 1}$

- Si: $\theta \in \mathrm{IIC}$

  $\mathbf{-1 < Cos\theta < 0}$

- Si: $\theta \in \mathrm{IIIC}$

  $\mathbf{-1 < Cos\theta < 0}$

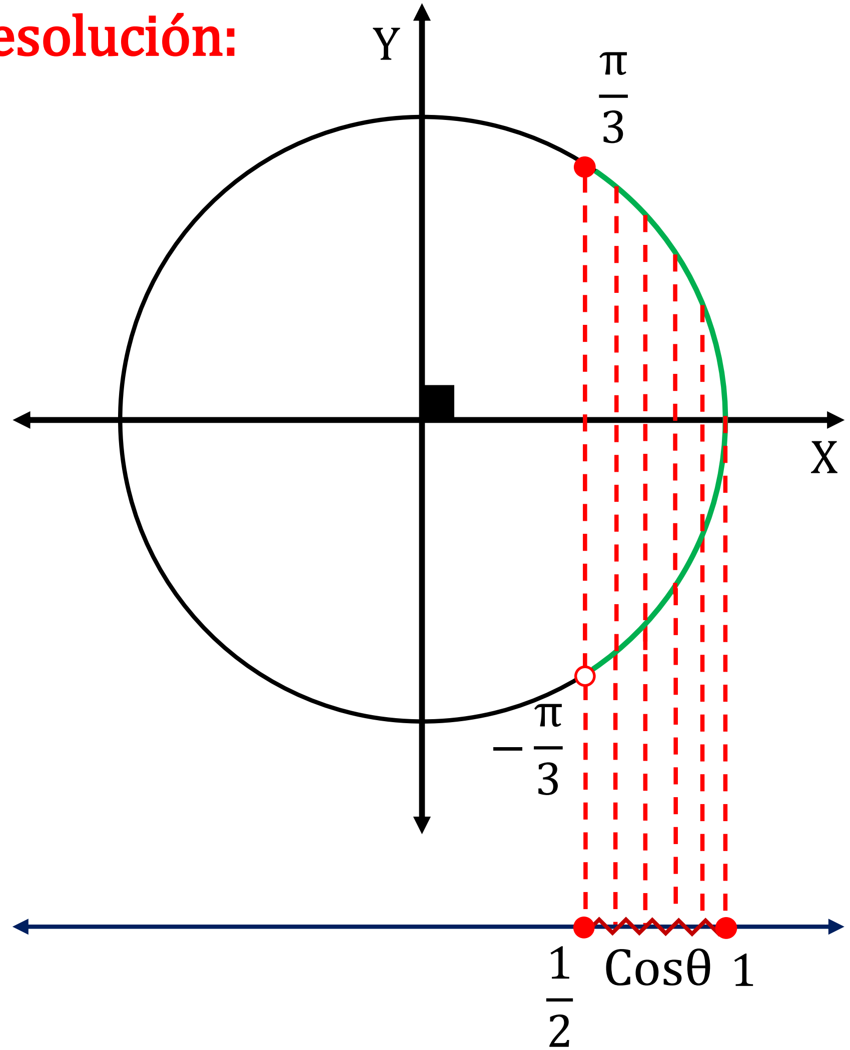
- Si: $\theta \in \mathrm{IVC}$

  $\mathbf{0 < Cos\theta < 1}$

Ejemplo:

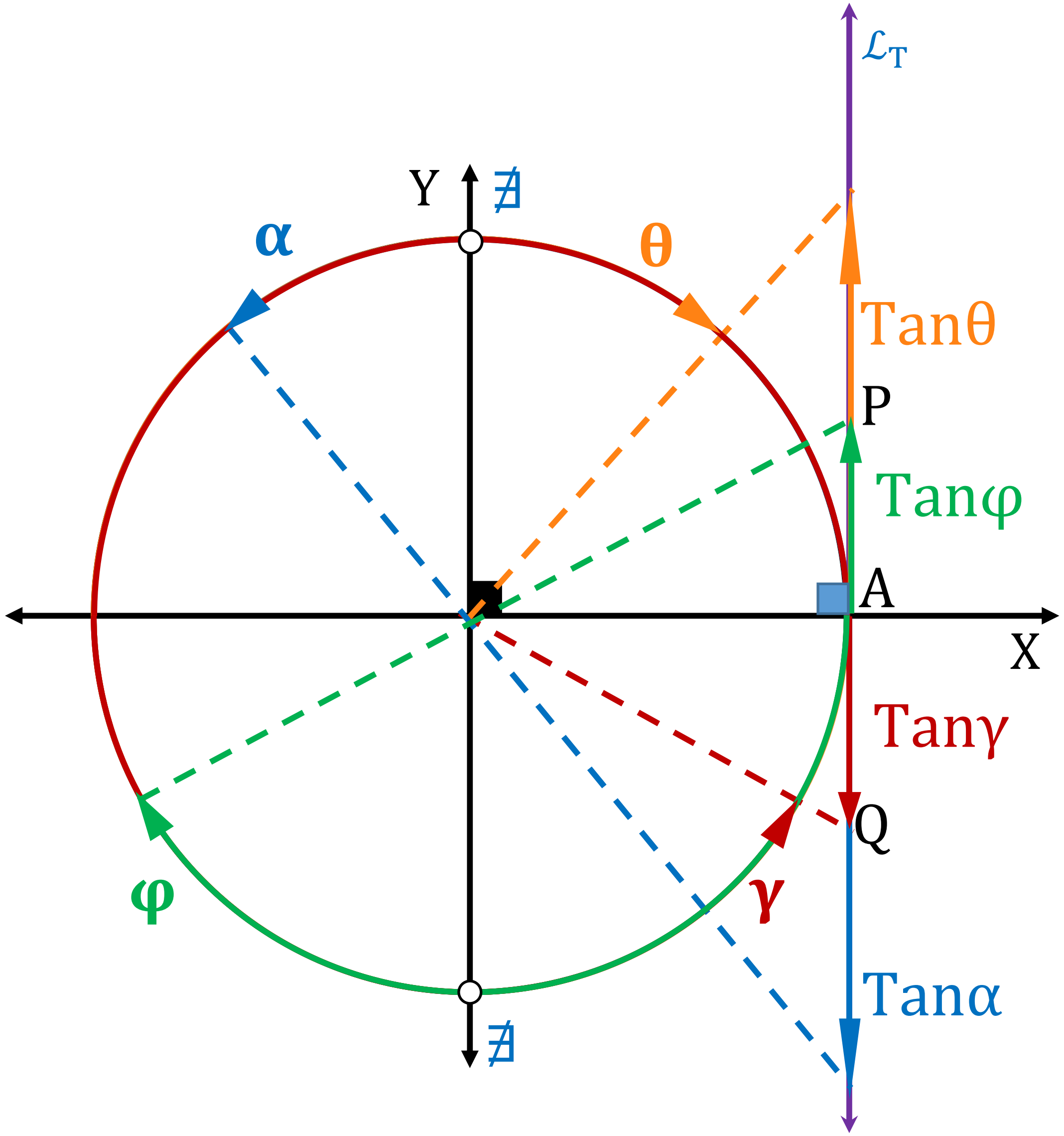
Determinar la extensión del Cosθ si $\theta \in \left]-\frac{\pi}{3};\frac{\pi}{3}\right]$

Resolución:



$\therefore \ \mathbf{Cos\theta} \in \left[\frac{1}{2};1\right]$

4.3) Representación de la línea Tangente:



✓ $\theta < \varphi < \alpha < \gamma$

✓ $\tan\alpha < \tan\gamma < \tan\varphi < \tan\theta$

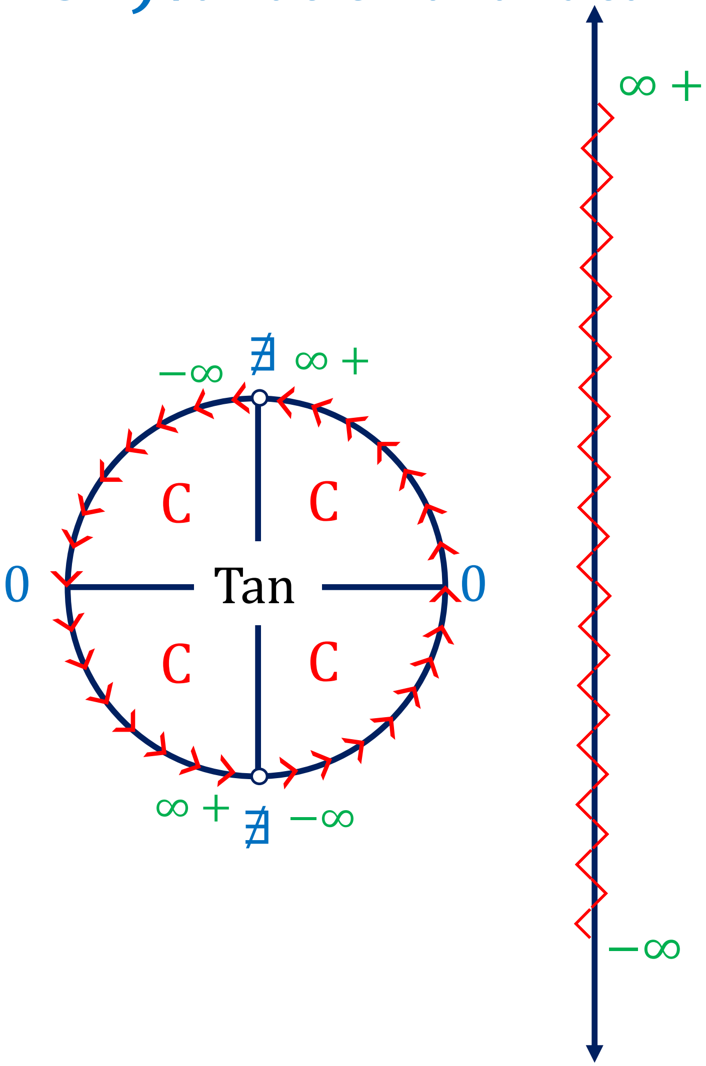
i. Valor $\begin{cases} (+) : \tan\theta \\ (-) : \tan\alpha \\ 0 \;\; : \tan\pi \end{cases}$

ii. Longitud: $(+)$, $|\text{Tangente}|$

$\overline{AP} = |\tan\varphi| = \tan\varphi$

$\overline{AQ} = |\tan\gamma| = -\tan\gamma$

4.3.1) Variación analítica :



i. Extensión:

$$\forall\, \theta \in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2} ; n \in \mathbb{Z}\right\}$$

$$-\infty < \text{Tan}\theta < \infty +$$

- Si: $\theta \in$ IC

  $\mathbf{0 < Tan\theta}$

- Si: $\theta \in$ IIC

  $\mathbf{Tan\theta < 0}$

- Si: $\theta \in$ IIIC

  $\mathbf{0 < Tan\theta}$

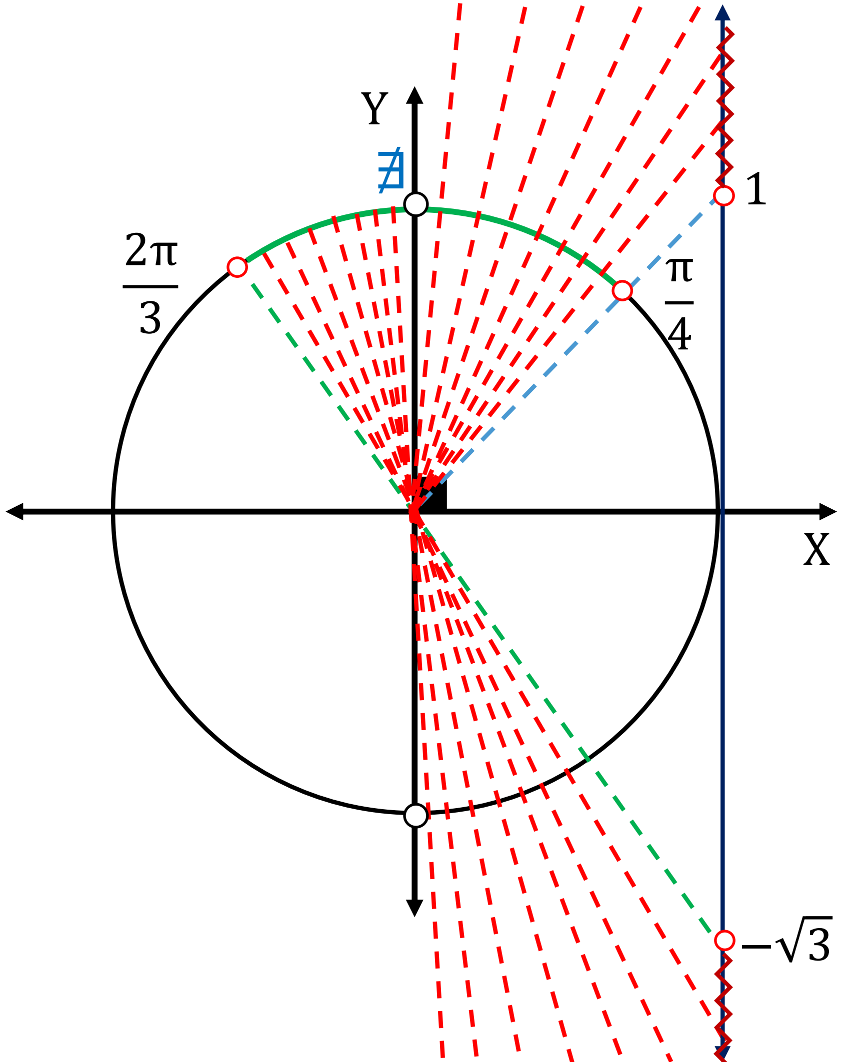
- Si: $\theta \in$ IVC

  $\mathbf{Tan\theta < 0}$

Ejemplo:

Determinar la extensión de la Tanθ si $\theta \in \left]\frac{\pi}{4};\frac{2\pi}{3}\right[$

Resolución:



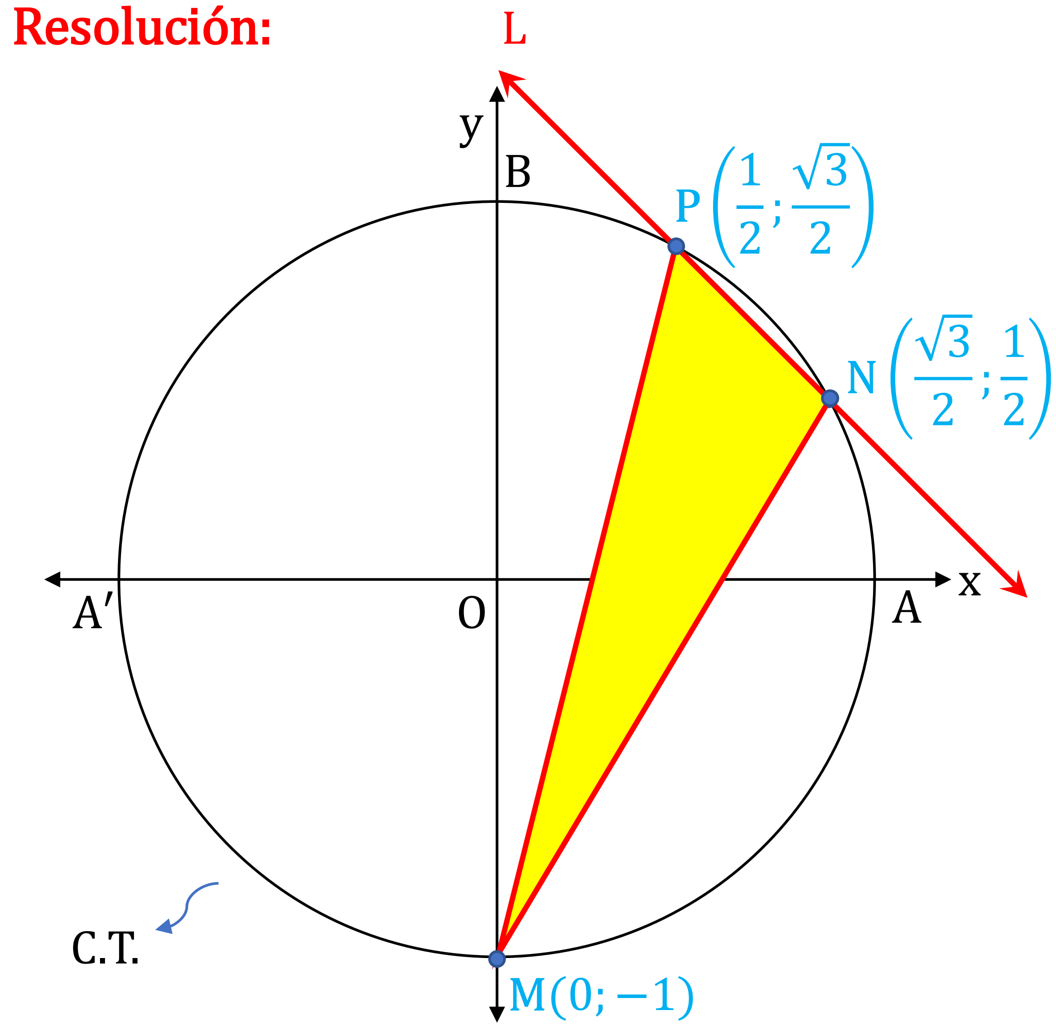
$\therefore\ \mathbf{Tan\theta} \in \left]-\infty;-\sqrt{\mathbf{3}}\right[\cup \left]\mathbf{1};+\infty\right[$

MOMENTO DE PRACTICAR

PROBLEMAS Y RESOLUCIÓN

1. En la figura se tiene que la ecuación de la recta es: $L: x+y=\dfrac{\sqrt{3}+1}{2}$ hallar el área de la región sombreada

Resolución:



$$\text{N y P}: \{L \cap C\} \begin{cases} L: y = \dfrac{\sqrt{3}+1}{2} - x \\ CT: x^2 + y^2 = 1 \end{cases}$$

$$x^2 + \left(\frac{\sqrt{3}+1}{2} - x\right)^2 = 1$$

$$2x^2 - (\sqrt{3}+1)x + 1 + \frac{\sqrt{3}}{2} = 1$$

$$2x^2 - (\sqrt{3}+1)x + \frac{\sqrt{3}}{2} = 0$$

$$\begin{array}{lll} 2x & -1 & \rightarrow x = \dfrac{1}{2} \\ x & -\dfrac{\sqrt{3}}{2} & \rightarrow x = \dfrac{\sqrt{3}}{2} \end{array}$$

$$\begin{array}{r|cc|l} & 0 & -1 & \\ -\dfrac{\sqrt{3}}{2} & \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} & 0 \\ \dfrac{1}{4} & \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} & \dfrac{3}{4} \\ 0 & 0 & -1 & -\dfrac{1}{2} \\ \hline \dfrac{1}{4} - \dfrac{\sqrt{3}}{2} & & & \dfrac{1}{4} \end{array}$$

$$S = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \mathbf{S = \frac{\sqrt{3}}{4}}$$

CLAVE: B

2. Hallar las coordenadas del punto P

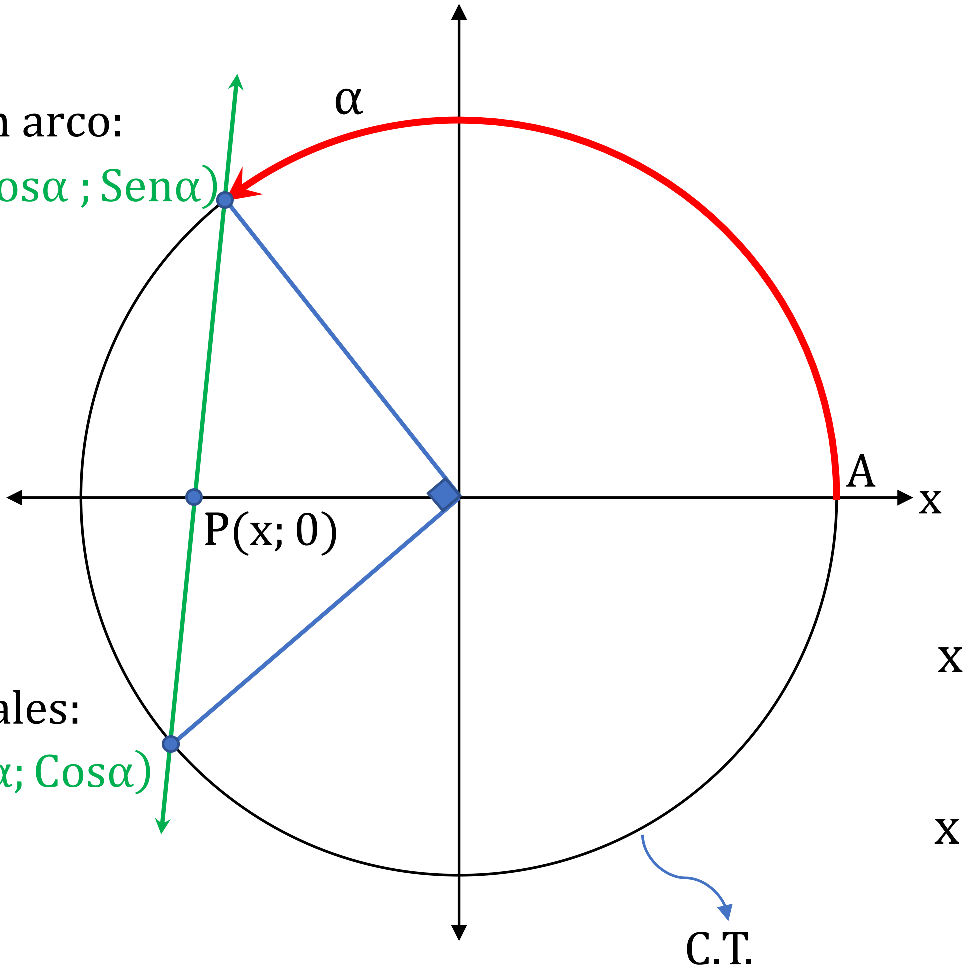
Resolución:

- Extremo de un arco:

  $M(\text{Cos}\alpha\,;\text{Sen}\alpha)$

- Coordenadas Ortogonales:

  $N(-\text{Sen}\alpha;\text{Cos}\alpha)$



$$m_{M\ y\ N} = m_{P\ y\ N}$$

$$\frac{\text{Sen}\alpha - \text{Cos}\alpha}{\text{Cos}\alpha - (-\text{Sen}\alpha)} = \frac{0 - \text{Cos}\alpha}{x - (-\text{Sen}\alpha)}$$

$$x + \text{Sen}\alpha = \frac{-\text{Cos}\alpha(\text{Cos}\alpha + \text{Sen}\alpha)}{\text{Sen}\alpha - \text{Cos}\alpha}$$

$$x = \frac{-\text{Cos}^2\alpha - \text{Sen}\alpha\text{Cos}\alpha}{\text{Sen}\alpha - \text{Cos}\alpha} - \text{Sen}\alpha$$

$$x = \frac{-\text{Cos}^2\alpha - \cancel{\text{Sen}\alpha\text{Cos}\alpha} - \text{Sen}^2\alpha + \cancel{\text{Sen}\alpha\text{Cos}\alpha}}{\text{Sen}\alpha - \text{Cos}\alpha}$$

$$x = \frac{-1}{\text{Sen}\alpha - \text{Cos}\alpha}$$

$$\therefore \mathbf{x = \frac{1}{\text{Cos}\alpha - \text{Sen}\alpha}}$$

CLAVE: D

3. Hallar la extensión de: $y = \text{Sen}^2x + 2|\text{Senx}|$

Resolución:

$$y = |\text{Senx}|^2 + 2|\text{Senx}| + 1 - 1$$

$$y = (|\text{Senx}| + 1)^2 - 1$$

$$x \in \mathbb{R}$$

$$-1 \leq \text{Senx} \leq 1$$

$$0 \leq |\text{Senx}| \leq 1$$

$$1 \leq |\text{Senx}| + 1 \leq 2$$

$$1 \leq (|\text{Senx}| + 1)^2 \leq 4$$

$$0 \leq \underbrace{(|\text{Senx}| + 1)^2 - 1}_{y} \leq 3$$

$$\therefore \mathbf{y} \in [\mathbf{0}; \mathbf{3}]$$

CLAVE: E

4. Halle la variación de "θ" en el segundo cuadrante para el cual se tiene : $\sqrt{3}\mathrm{Sen}^2x = 2\mathrm{Sen}\left(\theta + \frac{\pi}{6}\right)$

Resolución:

$$\frac{\pi}{2} < \theta < \pi$$

$$x \in \mathbb{R}$$

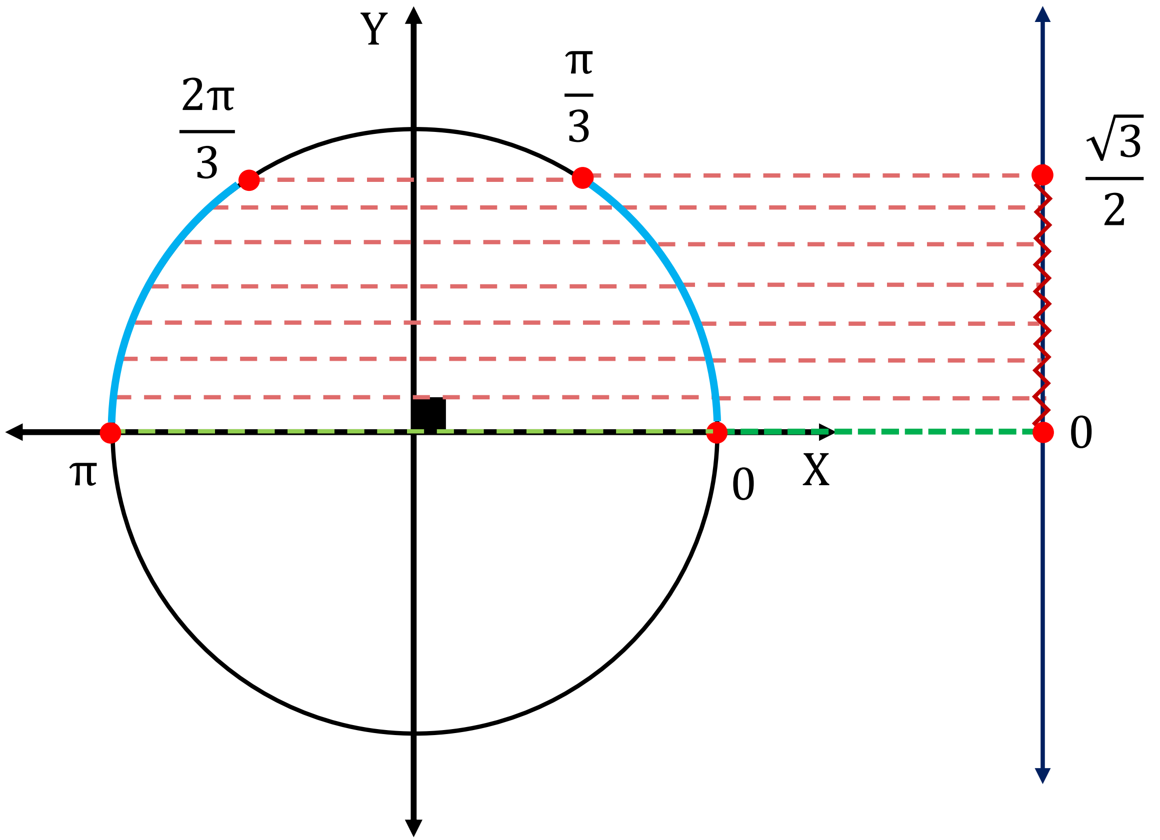
$$-1 \leq \mathrm{Sen}x \leq 1$$

$$0 \leq \mathrm{Sen}^2x \leq 1$$

$$0 \leq \sqrt{3}\mathrm{Sen}^2x \leq \sqrt{3}$$

$$0 \leq 2\mathrm{Sen}\left(\theta + \frac{\pi}{6}\right) \leq \sqrt{3}$$

$$0 \leq \mathrm{Sen}\left(\theta + \frac{\pi}{6}\right) \leq \frac{\sqrt{3}}{2}$$



$$0 \leq \theta + \frac{\pi}{6} \leq \frac{\pi}{3}$$

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$$

$$\frac{2\pi}{3} \leq \theta + \frac{\pi}{6} \leq \pi$$

$$\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$$

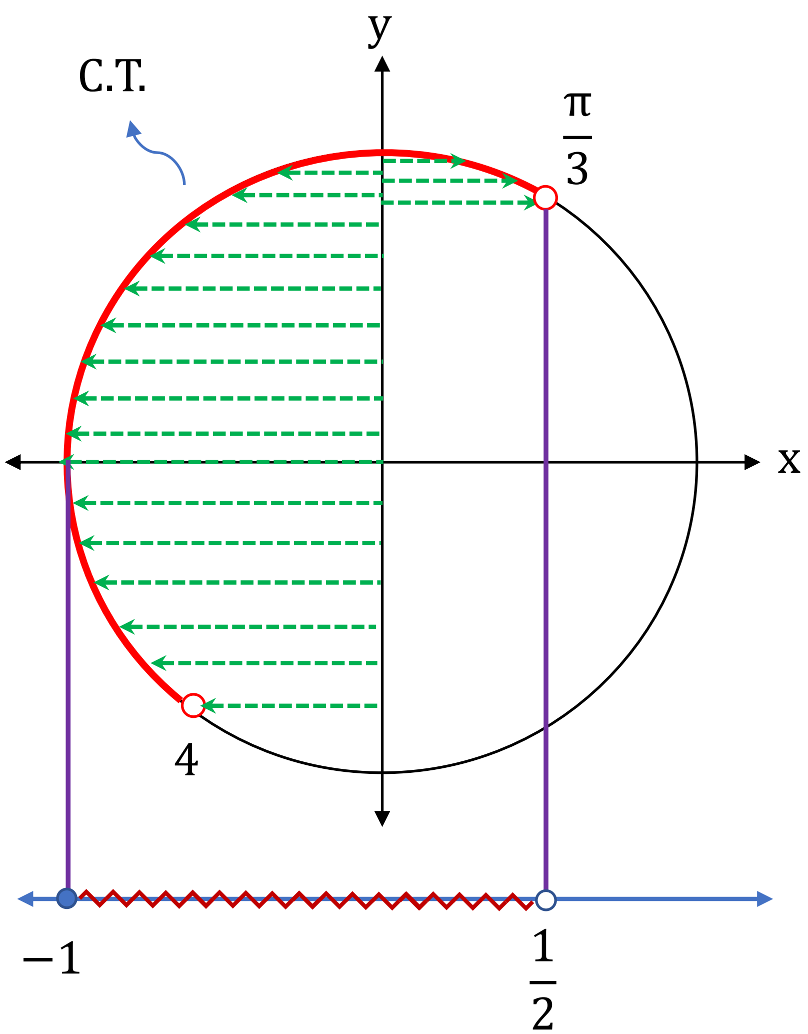
$$\boldsymbol{\frac{\pi}{2} < \theta \leq \frac{5\pi}{6}}$$

CLAVE: C

5. Determinar la variación de la siguiente expresión: $E=3+\cos^2\theta$, sabiendo que $\theta \in \left\langle \frac{\pi}{3}; 4 \right\rangle$

Resolución:

$$\frac{\pi}{3} < \theta < 4$$



$$-1 \leq \cos\theta < \frac{1}{2}$$

$$0 \leq \cos^2\theta < 1$$

$$3 \leq 3 + \cos^2\theta < 4$$

$$\mathbf{E \in [3; 4[}$$

CLAVE: B

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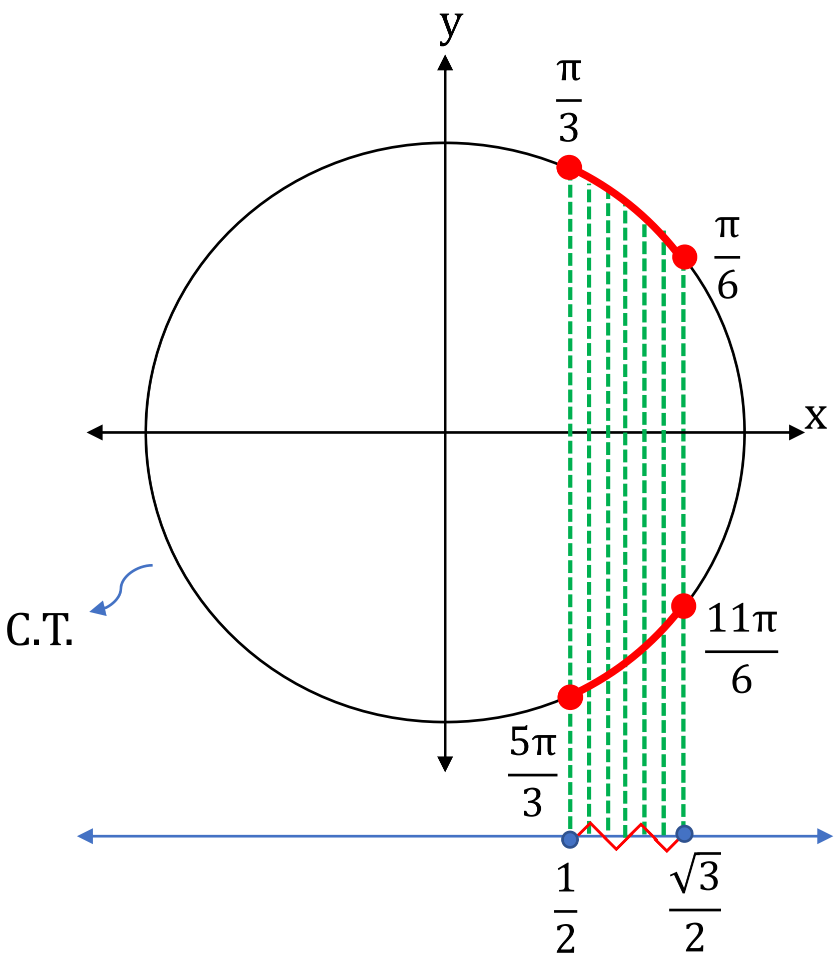
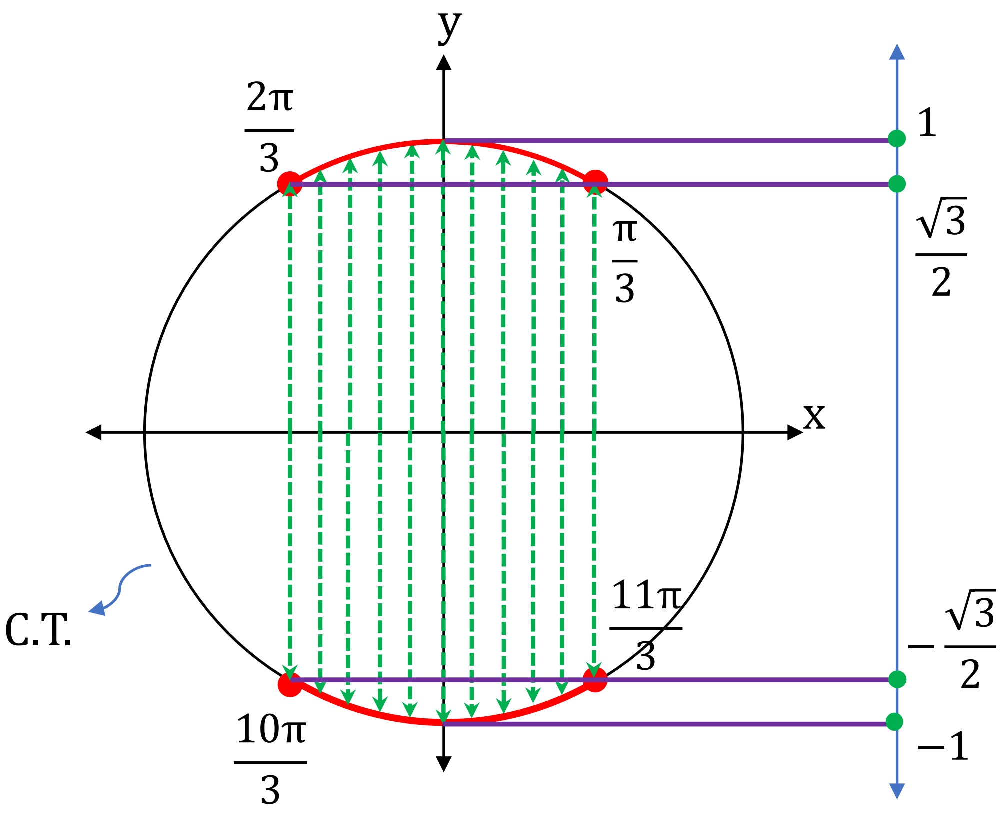
6. Si $\theta \in \langle 0; 2\pi]$; además se cumple: $\frac{1}{2} \leq \text{Cos}\theta \leq \frac{\sqrt{3}}{2}$, hallar la variación de "Sen2θ"

Resolución:

$$\frac{1}{2} \leq \text{Cos}\theta \leq \frac{\sqrt{3}}{2}$$

$$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3} \cup \frac{5\pi}{3} \leq \theta \leq \frac{11\pi}{6} \xrightarrow{\times 2} \frac{\pi}{3} \leq 2\theta \leq \frac{2\pi}{3} \cup \frac{10\pi}{3} \leq 2\theta \leq \frac{11\pi}{3}$$





$$\left[-1; -\frac{\sqrt{3}}{2}\right] \cup \left[\frac{\sqrt{3}}{2}; 1\right]$$

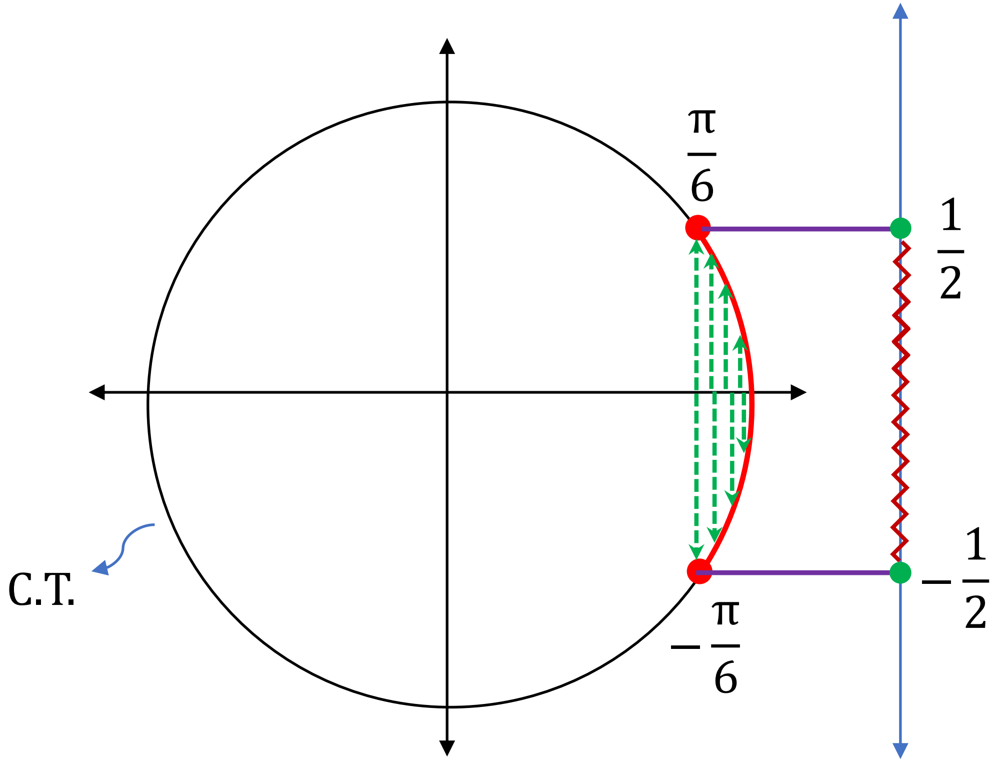
$$-1 \leq \text{Sen}2\theta \leq -\frac{\sqrt{3}}{2} \cup \frac{\sqrt{3}}{2} \leq \text{Sen}2\theta \leq 1$$

CLAVE: E

7. Si : $x \in <0; 2\pi>$; además: $\text{Tan}x = \sqrt{2\text{Sen}\theta + 2}$, hallar la variación de x, para $\theta \in \left[-\frac{\pi}{6}; \frac{\pi}{6}\right]$

Resolución:

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$$



$$-\frac{1}{2} \leq \text{Sen}\theta \leq \frac{1}{2}$$

$$-1 \leq 2\text{Sen}\theta \leq 1$$

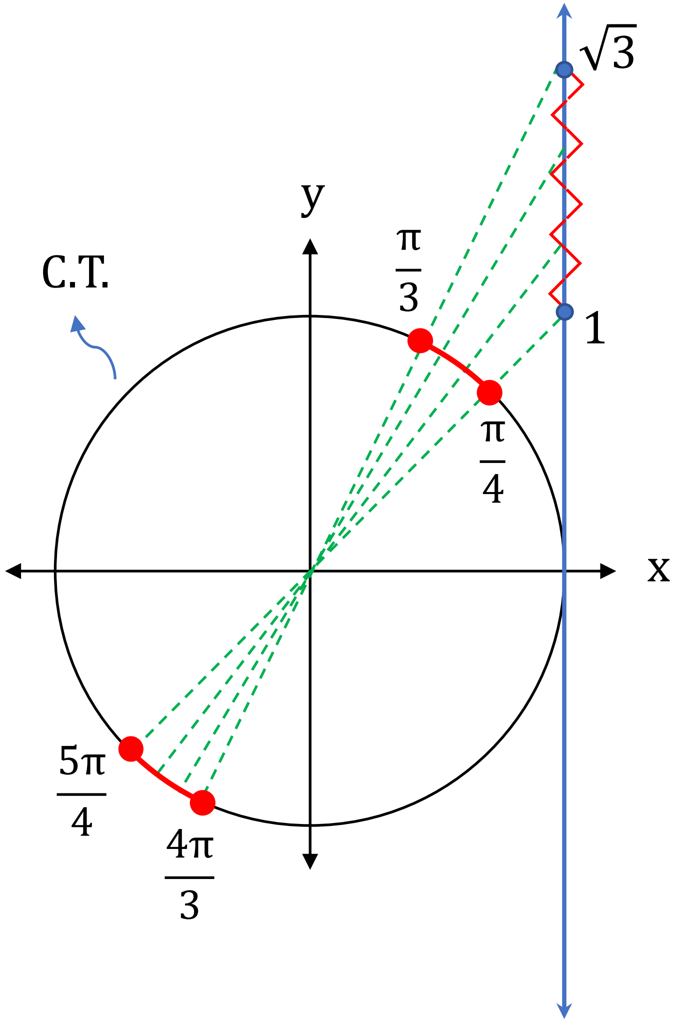
$$1 \leq 2\text{Sen}\theta + 2 \leq 3$$

$$\sqrt{1} \leq \sqrt{2\text{Sen}\theta + 2} \leq \sqrt{3}$$

$$1 \leq \text{Tan}x \leq \sqrt{3}$$

$$\frac{\pi}{4} \leq x \leq \frac{\pi}{3} \cup \frac{5\pi}{4} \leq x \leq \frac{4\pi}{3}$$

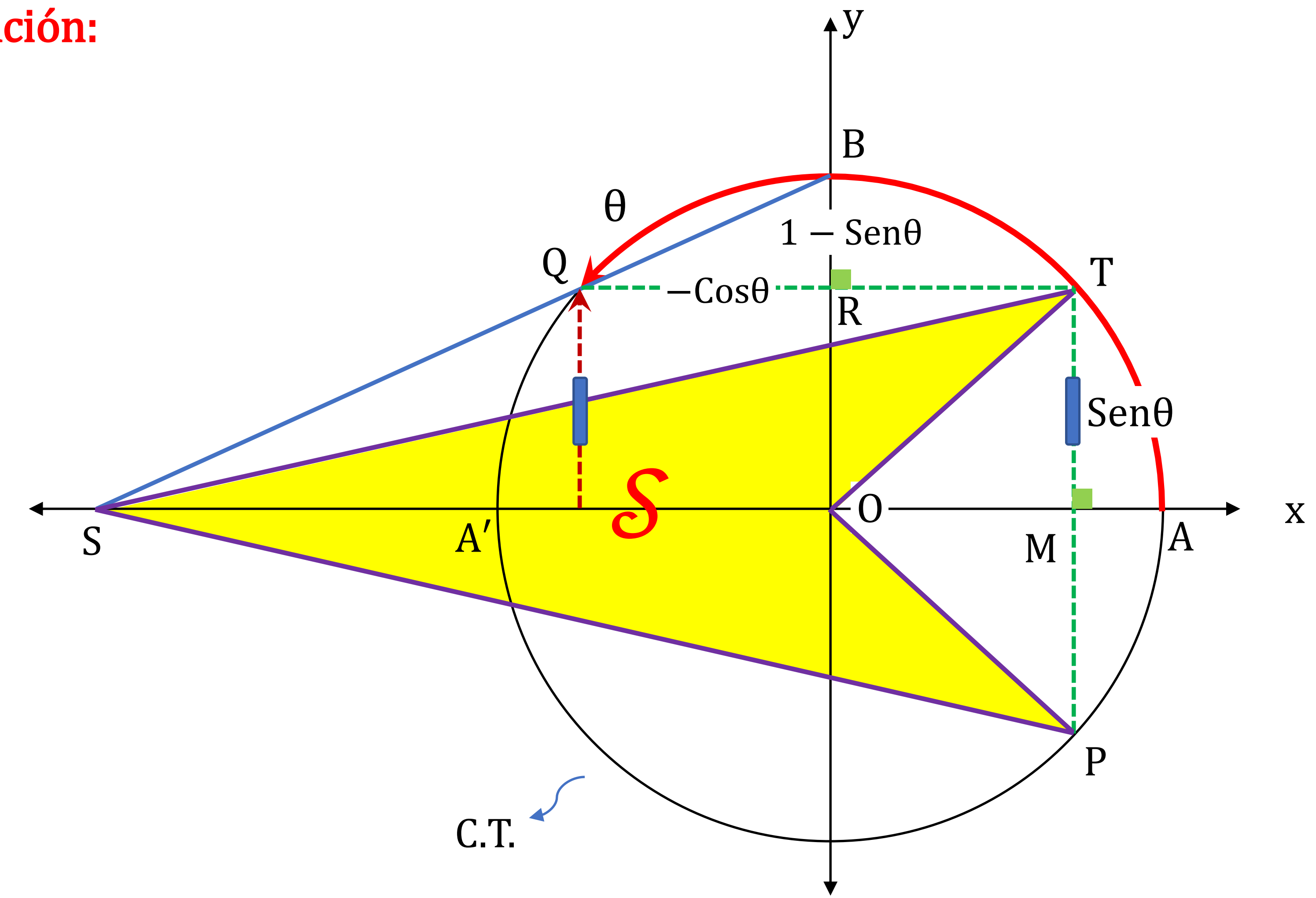
$$\therefore \mathbf{x} \in \left[\frac{\pi}{4}; \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{4}; \frac{4\pi}{3}\right]$$



CLAVE: C

8. Hallar el área de la región sombreada en términos de θ

Resolución:



$$\mathcal{S} = SO \times TM$$

$$\triangle BOS \sim \triangle BRQ$$

$$\frac{SO}{1} = \frac{-Cos\theta}{1 - Sen\theta}$$

$$SO = \frac{Cos\theta}{Sen\theta - 1}$$

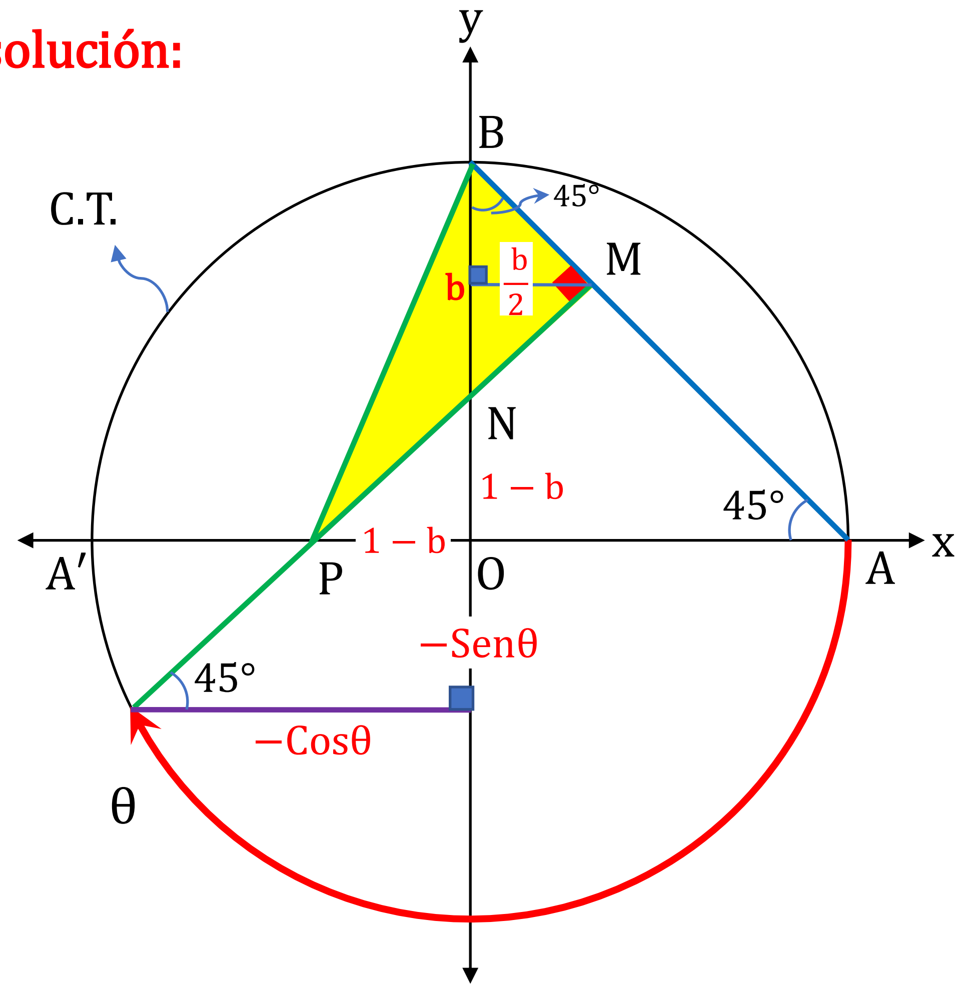
$$\mathcal{S} = \frac{Cos\theta}{Sen\theta - 1} \times Sen\theta$$

$$\therefore \mathcal{S} = \frac{\mathbf{Sen\theta Cos\theta}}{\mathbf{Sen\theta - 1}}$$

CLAVE: E

9. Hallar el área sombreada

Resolución:



$$-\text{Cos}\theta = 1 - b - \text{Sen}\theta$$

$$b = 1 - \text{Sen}\theta + \text{Cos}\theta$$

$$S_x = S_{\Delta PBN} + S_{\Delta MBN}$$

$$S_x = \frac{1}{2}b(1-b) + \frac{1}{2}b\left(\frac{b}{2}\right)$$

$$S_x = \frac{b}{2}\left(1 - b + \frac{b}{2}\right)$$

$$S_x = \frac{b}{2}\left(\frac{2-b}{2}\right)$$

$$S_x = \frac{(1 - \text{Sen}\theta + \text{Cos}\theta)(1 + \text{Sen}\theta - \text{Cos}\theta)}{4}$$

$$S_x = \frac{[1 + (\text{Sen}\theta - \text{Cos}\theta)][1 - (\text{Sen}\theta - \text{Cos}\theta)]}{4}$$

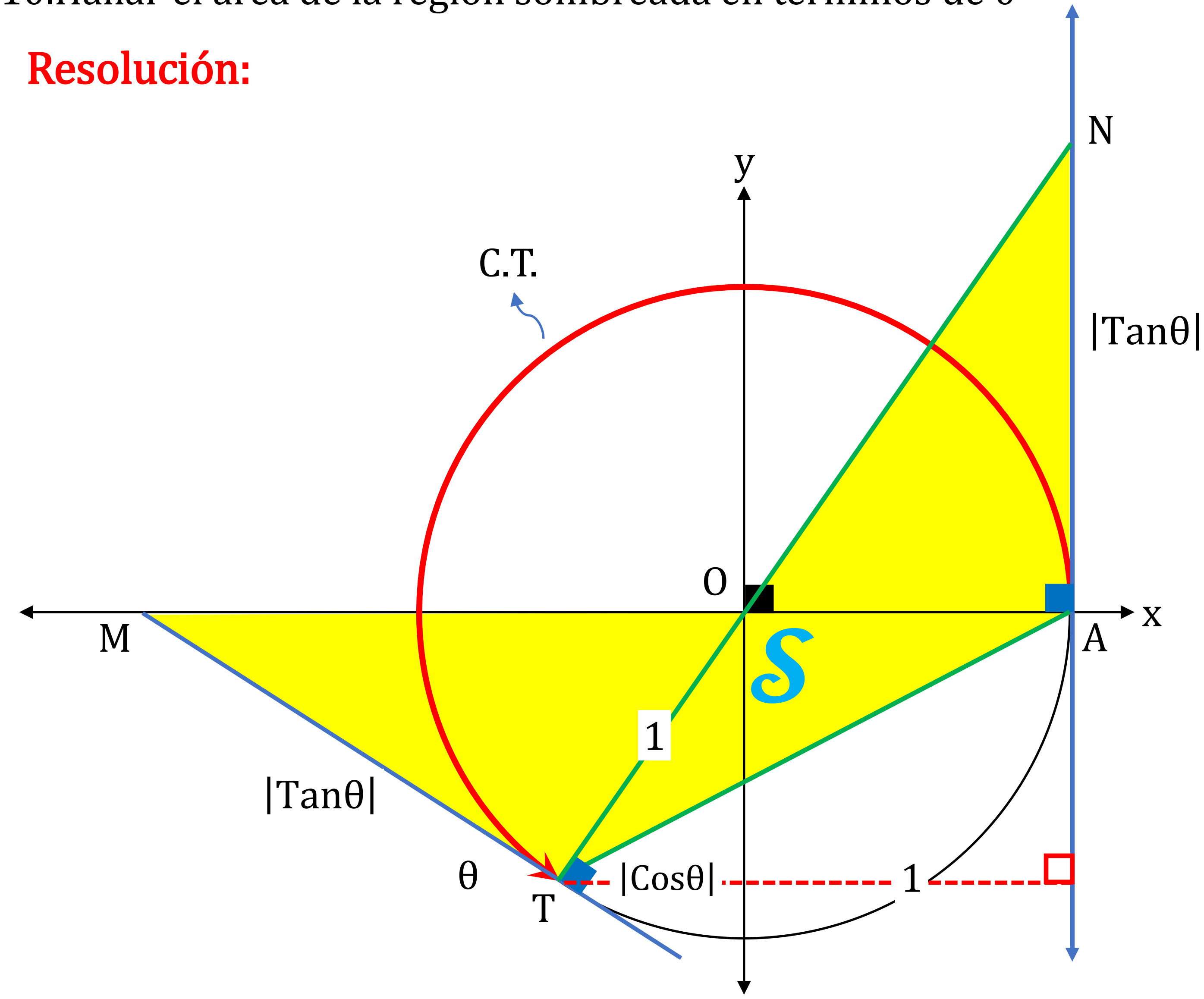
$$S_x = \frac{[1 - (\text{Sen}\theta - \text{Cos}\theta)^2]}{4}$$

$$\therefore S_x = \frac{\textbf{Sen}\theta\textbf{Cos}\theta}{2}$$

CLAVE: B

10.Hallar el área de la región sombreada en términos de θ

Resolución:



$$\mathcal{S} = S_{\Delta MTO} + S_{\Delta ANT}$$

$$\mathcal{S} = \frac{|\mathrm{Tan}\theta| \times 1}{2} + \frac{|\mathrm{Tan}\theta| \times (1 + |\mathrm{Cos}\theta|)}{2}$$

$$\mathcal{S} = \frac{|\mathrm{Tan}\theta|(2 + |\mathrm{Cos}\theta|)}{2}$$

$$\therefore \mathcal{S} = \frac{\mathbf{Tan\theta(2 - Cos\theta)}}{\mathbf{2}}$$

CLAVE: D

CIRCUNFERENCIA TRIGONOMÉTRICA

11. D
12. B
13. E
14. E
15. C
16. B
17. C
18. D
19. C
20. B

FIN DE LA SESIÓN

PRACTICA Y APRENDERÁS